Research Plan

Tatsuya Watanabe

1: Research on a non-homogeneous nonlinear elliptic equation

We consider the following nonlinear elliptic problem:

$$-\Delta u + u = g(u) + \lambda f(x), \ x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $f(x) \geq 0$ and $f(x) \not\equiv 0$. We know that if $\lambda > 0$ is small, then (1) has at least two positive solutions. Among them, one is characterized as a perturbation of the trivial solution to the unperturbed problem:

$$-\Delta u + u = g(u) \text{ in } \mathbb{R}^N. \tag{2}$$

Thus for small λ, we can see that the first solution is close to the unique solution of the linear problem: $-\Delta u + u = \lambda f(x)$. Hence we can know the shape of this solution.

On the other hand, we can expect that the second solution is a perturbation of a least energy solution of (2). My interest is to know the shape of the second solution. It is known that the least energy solution of (2) is positive, radially symmetric and radially decreasing with respect to a point. Since problem (2) is translation invariant, we can locate the maximum point of the least energy solution at any points. However the translation invariance gets broken at the moment λ becomes positive. Therefore we can expect that the location of the maximum point of the second solution should be determined by certain information of $f(x)$. If f is radially symmetric and radially decreasing with respect to a point, then we can see that any positive solutions of (1) should be radially symmetric and radially decreasing with respect to the same point by the moving plane method. In other words, we can say that the location of maximum points of all positive solutions of (1) are same as that of $f(x)$. In this point of view, I'm interested in which information of f does affect the location of the second solution when f fails to be either radially symmetric or radially decreasing. This problem still remains open even for the typical case $g(u) = |u|^{p-1}u$.

I also would like to mention that most previous works have considered the case $\lambda > 0$. By the bifurcation theory, we can see that a branch which bifurcates from a least energy solution of (2) extends to the range $\lambda < 0$. The question is whether this branch of solutions is a positive function. Since the least energy solution is positive, we can expect that its positivity does not get broken under a small perturbation. However when $\lambda < 0$, we can not apply the maximum principle, which we frequently use to obtain the positivity of solutions in elliptic problems. From this reason, the question above is still open although the expectation seems to be true. It seems to be difficult but interesting whether we can derive the positivity of solutions from their characterizations and so on, without using the maximum principle.

Problems like (1) with $\lambda < 0$ are said to be non-positone. They have been widely studied non-positone problems in bounded domains and obtained not only the existence of solutions but also symmetry breaking phenomena. On the other hand, there are only few results for non-positone problems in unbounded domains. From this point of view, this research seems to be interesting and challenging for me.

2: Research on solutions with a vortex to nonlinear Schrödinger equations in $\mathbb{R}^2$

My particular interest is whether there is a solution of the following problem:

$$-\epsilon^2 \Delta u + \left(\frac{\epsilon^2 n^2}{|x-a|^2} + \frac{\epsilon^2 m^2}{|x-b|^2} + V(x)\right)u = f(u), x \in \mathbb{R}^2$$

which has vortices at both a and b and its asymptotic behavior does depend on n and m. In my previous research, I obtained asymptotic profiles of solutions by analyzing the corresponding ODE.

However the above situation does not allow us to rewrite the problem to an ODE. Therefore we need new ideas for the analysis. As a first step, I would like to consider the following nonlinear Schrödinger equation in $\mathbb{R}^2$:

$$-\epsilon^2 \Delta u + (\frac{\epsilon^2 n^2}{|x|^2} + V(x))u = f(u), x \in \mathbb{R}^2$$

when the potential $V(x)$ is non-radial. Even though the potential is non-radial, it seems to be reasonable that a mountain pass solution u_ϵ does satisfy $u_\epsilon(x) \sim |x|^n$ near the origin as well as the radial case. I believe it is the most important step to prove this in the process of this research.

Finally I recently studied a fourth order elliptic problem. As a related topic, I'm interested in the Paneitz-Branson operator which is a fourth order version of the Laplace-Beltrami operator, and the corresponding Yamabe type problem. Especially I would like to know qualitative properties of solutions. In the study of fourth order elliptic problems, we face a difficulty because of the lack of general maximum principle. Actually under various situations, they have been intensively studied both proves and counterexamples of the maximum principle and related topics for higher order elliptic problems. However many unsolved problems still remain open. The maximum principle and the comparison principle are the most important tools to know both qualitative and quantitative properties of solutions in elliptic problems. Therefore I also would like to study the maximum principle, the comparison principle and linear eigenvalue problems for higher order elliptic operators as well as nonlinear problems.