Plan of the Research

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I'd like to continue to study the topology of the space $\mathcal{K}_{n,j}$ of long j-knots in $\mathbb{R}^n$, in particular its structure of iterated loop space as well as other related topics. I address (1) and (2) below for a time , and (3) in the future.

(1) On $\mathcal{K}_n = \mathcal{K}_{n,1}$. I pay attention to the operad action [2] and configuration space integral [4] because of their importance in the research of the topology of $\mathcal{K}_n$ from the homological viewpoint. I'd like to place the following conjecture at one of my goals.

Conjecture. (1) $H_*(\mathcal{K}_n)$ is a free Poisson algebra when $n > 3$.
(2) $H^*_{DR}(\mathcal{K}_n)$ can be completely described by graph cohomology classes via configuration space integrals.

The importance of this conjecture lies in topology and combinatorics; (1) would give us information on the homotopy type of a double loop space $\mathcal{K}_n$, and (2) would provide us a geometric proof of E^1-degeneracy (over $\mathbb{R}$) of Vassiliev spectral sequence. We can tackle these two statements simultaneously via Kronecker product, since each statement relates to nontriviality of (co)homology classes. I'd like to define a co-Poisson algebra structure on the level of graph complexes, which should be also defined on $H^*_{DR}(\mathcal{K}_n)$, and using them compute the Kronecker product inductively on the degrees. A method of my computation in [6] would be applied when we actually compute the Kronecker product. I would also make use of combinatorics of graphs and the related (co-)Poisson structure on the Hochschild complex [8].

(2) Generalizations for $\mathcal{K}_{n,j}$. Similar construction as those in (1) can be applied to $\mathcal{K}_{n,j}$. But such a study has not been progressed compared to the case of $\mathcal{K}_{n,1}$, though there are a few researches [1, 9]. I'd like to establish a foothold to study the iterated loop space structure of $\mathcal{K}_{n,j}$, by constructing 'algebraic model' which describes $H^*(\mathcal{K}_{n,j})$ via configuration space integral and 'Goodwillie calculus' [7]. It is still subtle what kind of graphs we should consider. So I'd like to seek suitable graphs through the constructions of classical knot invariants such as the Haefliger invariant, as was done in [5].

These constructions would become important, not only when studying the topology of $\mathcal{K}_{n,j}$ itself, but when considering their connections to that of $\mathcal{K}_{n,1}$. In [2] some maps such as R. Litherland's "deform-spun" map $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ (where Ω denotes the based loop space functor) and "suspension map" $\Sigma\mathcal{K}_{n,j} \to \mathcal{K}_{n+j,j}$ have been defined and some of their properties have been studied. The former map is shown to be compatible with the operad action. Via these maps, $\mathcal{K}_{n,1}$ might bring some information on $\mathcal{K}_{n,j}$. To study such relations algebraically, I plan to define some "algebraic model" for $\mathcal{K}_{n,j}$ such as graph complexes and corresponding maps on the model.

(3) Relations to low dimensional topology. The cohomology classes obtained from trivalent graph cocycles via configuration space integral correspond to the finite type invariants for knots in 3-space or 3-manifolds. But it has not been understood to which objects nontrivalent graphs relate. In [3] it was shown that the first homology of a connected component of $\mathcal{K}_3$ is trivial if and only if the corresponding knot type is trivial. In similar way I'd like to study the properties of knots which the existence of the cohomology classes arising from nontrivalent graphs obstruct. To consider this problem I would prepare as many elements of $H^*_{DR}(\mathcal{K}_n)$ from nontrivalent graphs as possible.

As for $\pi_*(\mathcal{K}_n)$, as a consequence of [7], a spectral sequence can be constructed by using the homotopy groups of configuration spaces $\mathrm{Conf}(\mathbb{R}^n, k)$. The Lie algebra of $\pi_*\mathrm{Conf}(\mathbb{R}^n, k)$ is known to coincide with that associated with the lower central series of pure braid groups. It is also known that the Lie algebra can be regarded as a (co)simplicial group and relates to Vassiliev invariants for pure braids and the homotopy groups of spheres and so on. I plan to study the relationship between these objects and the topology of $\mathcal{K}_n$ via the spectral sequence. For instance, I'd like to consider the properties of the spectral sequence and try to compute them explicitly, using the topology of the geometric realization of the above simplicial group.

References

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