Summary of research

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(1) Abstract. A long j-knot in $\mathbb{R}^n$ is an embedding $f : \mathbb{R}^j \hookrightarrow \mathbb{R}^n$ with fixed behaviour at infinity. We equip the set $\mathcal{K}_{n,j}$ of long j-knots in $\mathbb{R}^n$ with C^∞-topology. My research interest is in the topology of the space $\mathcal{K}_{n,j}$.

Among various approaches to $\mathcal{K}_{n,j}$, such as singularity-theoretical one by V. Vassiliev and homotopy-theoretical one by means of 'Goodwillie calculus,' I focus on that which regards $\mathcal{K}_{n,j}$ as an iterated loop space via an action of operad [1, 8], and the method arising from perturbative Chern-Simons theory, to describe $H^*(\mathcal{K}_{n,j})$ in terms of graphs. So far [3, 4, 5, 6, 7] I have (i) proved nontriviality of the Poisson structure on $H_*(\mathcal{K}_{n,1})$ induced by the homotopy type of $\mathcal{K}_{n,1}$ as a double loop space, (ii) constructed the first example of nonvanishing cohomology class by using nontrivalent graphs, and (iii) generalized the graphical description of $H^*(\mathcal{K}_{n,j})$ to the case of arbitrary n, j. Details are as follows.

(2) Results.

(2-1) As shown in [1], the **little** $(j+1)$**-balls operad** acts on the space of 'framed long knots' $\widetilde{\mathcal{K}}_{n,j}$. A connected space which admits such an action has a homotopy type of a $(j+1)$-fold loop space, and on its homology group j**-Poisson bracket** λ_j is defined. Here j-Poisson bracket is a graded Lie bracket satisfying the Leibniz rule;

$$\lambda_j : H_p(\widetilde{\mathcal{K}}_{n,j}) \otimes H_q(\widetilde{\mathcal{K}}_{n,j}) \longrightarrow H_{p+q+j}(\widetilde{\mathcal{K}}_{n,j}), \quad \lambda_j(x, yz) = \lambda_j(x,y)z \pm y\lambda_j(x,z)$$

where the products yz and so on are induced by connected-sum of long knots. We can expect that we may understand the iterated loop space structure of $\widetilde{\mathcal{K}}_{n,j}$ by studying the Poisson algebra structure of $H_*(\widetilde{\mathcal{K}}_{n,j})$, but even the nontriviality of the Poisson bracket is not known except for the case of $\mathcal{K}_{3,1}$. Here is my first result.

Theorem 1 ([4])**.** When $n > 3$ is odd, there exist $x, y \in H_*(\widetilde{\mathcal{K}}_{n,1})$ such that $\lambda_1(x,y) \neq 0$. □

For the proof I made use of the description of $H^*_{DR}(\mathcal{K}_{n,1})$ in terms of graphs [2]. According to [2], to a graph Γ, a fibration $C_\Gamma \to \mathcal{K}_{n,1}$ with configuration spaces as its fibers and a differential form on C_Γ are assigned, and we can obtain an element $I(\Gamma) \in \Omega^*_{DR}(\mathcal{K}_{n,1})$ via an integration along the fiber. This correspondence I defines a cochain map from a graph complex to $\Omega^*_{DR}(\mathcal{K}_{n,1})$ when $n > 3$, and produces cohomology classes, which can be thought of as higher dimensional analogues of finite type invariants for knots, from trivalent graph cocycles. By using this framework, I constructed a dual cohomology class to $\lambda(x,y)$ in Theorem 1. This is the first notrivial example of a nontrivalent graph cocycle Γ, and proves simultaneously the following.

Corollary 2 ([4])**.** When $n > 3$ is odd, the map I is nontrivial for nontrivalent graph cocycles. □

As for the graph cocycle Γ above, it gives a nontrivial cohomology class even when $n = 3$.

Theorem 3 ([6])**.** $I(\Gamma) \in \Omega^1_{DR}(\mathcal{K}_{3,1})$ is a closed form. The integration of $I(\Gamma)$ over the "Gramain cycle" of each component of $\mathcal{K}_{3,1}$ is equal to Casson's invariant of long knots. □

(2-2) In [8], as an application of 'Goodwillie calculus,' an approximation of $\mathcal{K}_{n,1}$ by configuration spaces was given. As a corrolarry we have an action of little 2-balls operad on $\mathcal{K}_{n,1}$. On the other hand, this approximation provides us certain spectral sequence converging to $H_*(\mathcal{K}_{n,1})$, whose E^1-term coincides with the **Hochschild complex** [9] and admits an action of little 2-balls operad in an algebraic sense ('Deligne conjecture'). I compared these two actions of operads in the homological level, and obtained the following.

Theorem 4 ([3])**.** The above spectral sequence converges to $H_*(\mathcal{K}_{n,1})$ as a Poisson algebra. □

This gives us not only a topological interpretation of Deligne conjecture but also a method to compute the Poisson algebra structure on the spectral sequence. Theorem 1 is proved from this viewpoint.

(2-3) Most of the methods prepared to study $\mathcal{K}_{n,1}$ can be expected to be generalized to the cases of general $\mathcal{K}_{n,j}$. I generalized configuration space integrals mentioned in (2-1) following [10] and so on, and obtained the following.

Theorem 5 ([5, 6])**.** When $n > j \geq 3$ are odd, there exists a cochain map $I : \mathcal{D}^* \to \Omega^*_{DR}(\mathcal{K}_{n,j})$ from a graph complex $\mathcal{D}^*$ such that I produces nonzero elements of $H^{2k(n-j-2)}_{DR}(\mathcal{K}_{n,j})$ $(k \geq 1)$. In similar way we can construct an element of $H^{2n-3j-3}_{DR}(\mathcal{K}_{n,j})$ when $n - j \geq 3$ is odd, and in particular cases $2n - 3j - 3 = 0$, this coincides with the Haefliger invariant for $\mathcal{K}_{6k,4k-1}$. □

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