Summary of my research

Takehisa Tsujii

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ and $\mathfrak{g}$ the Lie algebra of G. In 1976, P. Bala and R. W. Carter proved the classification of nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ by using the $\mathfrak{sl}_2$-theory; this is called the Bala-Carter theorem. Afterward, K. Pommerening showed that the Bala-Carter theorem can be extended to good characteristic. This fact is a basic result in the representations of algebraic groups, and it is especially important in the theory of nilpotent orbits in good characteristic. For example, the following results is shown by Pommerening's theorem: the existence of cocharacters associated to the nilpotent elements in $\mathfrak{g}$, the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$, and the dimension theorem between the variety of Borel subalgebras containing X and the nilpotent orbits $\mathrm{Ad}(G)(X)$ for each nilpotent element X in $\mathfrak{g}$. However his proof needed case-by-case computations in each root system, and some details of these computations are omitted. So noncomputational proof was expected.

In 2003, A. Premet gave a conceptual proof of Pommerening's theorem by using the Kempf-Rousseau theory. At the same time, he proved the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$. However the proof is considerably difficult compared with proof of the Bala-Carter theorem; Premet's proof uses the invariant theory, the theory of finite reductive groups, and the Bala-Carter theorem.

So we thought whether there was more concise proof, and one idea was hit on; the idea is to show that for any distinguished nilpotent element X the optimal parabolic subgroup $P(X)$ is distinguished and the Richardson orbit corresponding to $P(X)$ contains X. It seems that the above-mentioned way is the most reasonable when thinking about proof that uses the Kempf-Rousseau theory, but various problems were expected to be caused in this way; two acute problems were caused.

- For each nilpotent element X in $\mathfrak{g}$, is there an optimal cocharacter for X such that X is a $m(X)$-weight vector for λ? (Here $m(X)$ is the certain positive integer determined by X.)
- For each nilpotent element X in $\mathfrak{g}$, do we have $m(X) \leq 2$?

If we use the proof of Premet, it is understand that these are both correct. Therefore anticipating that there should exist a direct proof of this result, and we have obtained an even simpler proof of Pommerening's theorem. We also simplified Premet's proof of the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$ [Theorem 4.11]. The main point that became concise is as follows.

- The Bala-Carter theorem need not be used.
- The theory of finite reductive groups need not be used.
- Because we only consider "distinguished" nilpotent elements, a part of proof has been simplified more.