A summary of my studies

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I have studied geometry of strongly pseudo-convex manifold (in abbreviation s.p.c. manifold), and obtained, as main results, the Serre duality theorem for holomorphic vector bundles over a complete s.p.c. manifold [1],[2], a continuity method for sub-elliptic Monge-Ampère equation [3], and a removable singularity theorem for J-holomorphic mappings between s.p.c. manifolds [4]. As examples of s.p.c. manifold, real hypersurfaces in a complex manifold and a unit sphere bundle over a manifold with constant curvature are well studied. Since on a s.p.c. manifold there exist a nondegenerate 2-form induced by a contact form and integrable complex structure J, it is well known that many analogies to symplectic or complex geometry hold. Hodge decomposition of their cohomology groups and $\partial\overline{\partial}$-lemma for Sasakian manifolds are famous. In particular, I have studied the geometry of s.p.c. manifolds from the view point of similarity to the even-dimensional geometry.

The Serre duality theorem of cohomology groups is a fundamental theorem in a complex geometry and is a foundation to calculate cohomologies by combining with vanishing theorems. Although an analogous theorem for a compact s.p.c. manifold with a trivial vector bundle had been proved by N.Tanaka, it is also important to prove this theorem for arbitrary vector bundles from an aspect of similarity with the complex geometry. The result in [2] is a generalization of this Serre duality theorem to a non-compact s.p.c. manifold. It is known that a s.p.c. manifold admits a distance called Carnot-Caratheodory distance and it is proved that, by my coauthor, Visiting Assistant Professor J.Masamune, the completeness with respect to this distance influences the symmetry of a Laplace operator over the s.p.c. manifold. Using these facts, we obtain the result in [2].

Since an almost complex structure J on a s.p.c. manifold satisfies an integrability condition, it is known that many analogies to complex geometry also hold. Especially, a geometry of Sasakian manifolds strongly is tied to a Kähler geometry by its coordinates which are transversal with respect to the contact distribution, because of the characteristic vector field is Killing when a manifold is Sasakian.

The Calabi conjecture is a classical problem in the Kähler geometry and is crucially related with the problem of existence of Sasaki-Einstein metric over Sasakian manifolds. It is known that an analogue problem of the case the underlying manifolds are Sasakian manifolds reduces to a problem in the Kähler geometry by an argument of transversal structure. On the other hand, our alternative proof which is obtained in [3] requires the condition to be Sasakian for only technical reasons.

In submitted paper [4], we investigate J-holomorphic mappings between strongly pseudo-convex manifolds. The theory of J-holomorphic curves in symplectic geometry is initiated by M.Gromov, and brings fruitful results to study of global symplectic geometry, and I intend to construct an analogous theory. It was a crucial problem to make sure that the moduli space is a compact manifold in the symplectic geometry for constructing quantam cohomology. We need to observe limits of pseudo-holomorphic curves when we consider a compactification of the moduli space and a removable singularity theorem for J-holomorphic curves played an important role in this observation. When we deal with J-holomorphic mappings for s.p.c. manifolds, we moreover have to consider the limit when J-holomorphic mappings move along orbit of characteristic vector fields. In [4], I showed the movement of this direction can be controlled by parameter transformation of the domain of mappings, at least in the case that the domain is a 3-dimensional Sasakian manifold.