Research Results

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Background of researches

To estimate the number of holomorphic sections of a given holomorphic family (M, π, D) is a fundamental problem. Here s is called to be a *holomorphic section* of (M, π, D) if s is a holomorphic map of a Riemann surface D into a two-dimensional complex manifold M and the composed map $\pi \circ s$ is the identity map of D. Denote by $\mathcal{S}$ the set of all holomorphic sections of (M, π, D).

Let C be a Riemann surface with a fixed-point-free involution τ and $f : D \to C$ an unbranched covering C. Assume that the genus $g(C)$ of C is $g(C) \geqq 2$. M. F. Atiyah constructed a two-sheeted branched covering $\Pi : M \to D \times C$ of the product $D \times C$ branched over two graphs of f and $\tau \circ f$ in $D \times C$. Here M is a two-dimensional complex manifold. We define π the composed map of $\Pi : M \to D \times C$ and the projection $D \times C \to D$, then the triple (M, π, D) becomes a holomorphic family of Riemann surfaces.

Since $g(C) \geqq 2$, we obtain the number $\sharp\mathcal{S}$ of all elements of $\mathcal{S}$ as follows. We define π' to be the composite of $\Pi : M \to D \times C$ and the projection $D \times C \to C$. For an element $s \in \mathcal{S}$, the composed map $\pi' \circ s$ of s and $\pi' : M \to C$ is a holomorphic map from D to C. Setting $\pi'\mathcal{S} = \{\pi' \circ s \mid s \in \mathcal{S}\}$, we see that $\pi'\mathcal{S}$ is contained in $\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$, where $\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$ is the set of all non-constant holomorphic maps from D to C. Since $g(C) \geqq 2$, it is well known that $\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$ is finite, for example, by M. Tanabe. Hence we have an estimation of $\sharp\mathcal{S}$.

On the other hand, if $g(C) = 1$, then $\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$ is infinite. Thus it is difficult for me to estimate $\sharp\mathcal{S}$. Consequently it is important to study the estimation of $\sharp\mathcal{S}$ when $g(C) = 1$.

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Now let C be a torus and $f : D \to C \setminus \{0\}$ be a four-sheeted unbranched covering of $C \setminus \{0\}$ for a point 0 of C. Moreover we define the 0-map $0 : D \to C$ by $d \mapsto 0$.

In [2], we constructed a two-sheeted branched covering $\Pi : M \to D \times C$ of the product $D \times C$ branched over two graphs of f and the 0-map in $D \times C$. Denoting by π the composed map of $\Pi : M \to D \times C$ and the projection $D \times C \to D$, then we have a holomorphic family (M, π, D) of Riemann surfaces of genus two. In [2] we studied the family (M, π, D). So that we showed $\sharp\mathcal{S}$ is at most 10 in general.

Now, it is important to study how many distinct complex structures could be assigned on M. In [3] by using the theory of Teichmüller spaces, we showed there is at most one complex structure on M which makes (M, π, D) a holomorphic family.