Results of my research

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I study G_2-invariant geometrical objects of oriented 6-dimensional hypersurfaces and curves in $\mathrm{Im}\,\mathfrak{C}$.

Any orientable hypersurface of $\mathrm{Im}\,\mathfrak{C}$ has the special geometrical properties as follows. Let M^6 be an orientable (connected) 6-dimensional manifold and φ be an isometric immersion from M^6 to $\mathrm{Im}\,\mathfrak{C}$. The octonions is considered as a pair of the quaternions $\mathbf{H}\oplus\mathbf{H}$. We define the oriented basis (the orientation) of $\mathrm{Im}\,\mathfrak{C}$ as

$$\mathrm{Im}\,\mathfrak{C} = span_{\mathbf{R}}\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\},$$

where $\{i, j, k\}$ is the basis of pure imaginary part of quaternions and $\varepsilon = (0, 1) \in \mathbf{H}\oplus\mathbf{H}$. Then M^6 admits the orientation which is compatible with the above orientation of $\mathrm{Im}\,\mathfrak{C}$ such that

$$\xi \wedge T_p(M^6) = \mathrm{Im}\,\mathfrak{C},$$

where ξ is a unit normal vector field whole on M^6. We define the almost complex structure J_φ as

$$\varphi_*(J_\varphi X) = \varphi_*(X)\xi.$$

Then J_φ satisfies $J_\varphi^2 = -I$ and $g_\varphi(J_\varphi X, J_\varphi Y) = g_\varphi(X, Y)$ where g_φ denote the induced metric from the canonical metric of $\mathrm{Im}\,\mathfrak{C}$ and X, Y are vector fields on M^6. Therefore any orientable hypersurface M^6 of $\mathrm{Im}\,\mathfrak{C}$ admits the almost Hermitian structure. Moreover, by taking account of the algebraic properties of the octonions, the structure group of the tangent bundle of hypersurface of $\mathrm{Im}\,\mathfrak{C}$ reduces to the special unitary group $SU(3)$ of degree 3.

Let $\varphi : M^6 \to \mathrm{Im}\,\mathfrak{C}$ and $\varphi' : N^6 \to \mathrm{Im}\,\mathfrak{C}$ be two isometric immersions. We call φ and φ' are G_2(resp. $SO(7)$)-congruent if there exist a $g \in G_2$ (resp. $\in SO(7)$) and an orientation preserving diffeomorphism $\psi : M^6 \to N^6$ satisfying

$$g \circ \varphi = \varphi' \circ \psi$$

up to a parallel displacement. We can easily see that, if φ and φ' are G_2-congruent, then the two induced almost complex structures and the induced metrics coincide. Obviously, the G_2-congruency is an equivalent relation. However, in general, even if φ and φ' are $SO(7)$-congruent, but the induced almost complex structures are different.

We will classify almost complex structures of $M^6 = S^k \times \mathbf{R}^{6-k}$ into 4-types.

(1) The case of $M^6 = \mathbf{R}^6,\ S^1 \times \mathbf{R}^5,\ S^5 \times \mathbf{R}^1,\ S^6$, then the induced almost Hermitian structures are also unique under the action of $SO(7)$, (up to the action of G_2) and it acts transitively on M^6. The automorphism groups of the induced almost Hermitian

structures and the isometry groups of M^6 are as followings

M^6	$Aut(M^6, J, g)$	$Iso^+(M^6)$
$\mathbf{R}^6$	$\mathbf{R}^6 \rtimes SU(3)$	$\mathbf{R}^6 \rtimes SO(6)$
$S^1 \times \mathbf{R}^5$	$U(2) \ltimes \mathbf{R}^5$	$SO(2) \times (SO(5) \ltimes \mathbf{R}^5)$
$S^5 \times \mathbf{R}^1$	$SU(3) \times \mathbf{R}^1$	$SO(6) \times \mathbf{R}^1$
S^6	G_2	$SO(7)$

(2) The case of $M^6 = \mathbf{R}^2 \times S^4$, then the induced almost Hermitian structure is also unique under the action of $SO(7)$, (up to the action of G_2) and it does not act transitively on $\mathbf{R}^2 \times S^4$. The automorphism group of the induced almost Hermitian structure and the isometry group of $\mathbf{R}^2 \times S^4$ are $U(2) \ltimes \mathbf{R}^2 (\subset SO(5) \times (SO(2) \ltimes \mathbf{R}^2))$

(3) The case of $M^6 = S^2 \times \mathbf{R}^4$, then the induced almost complex structures of $g \circ \varphi$ are different from the original that of φ. If the parameter of deformation $\alpha \in (0, \pi/3)$, then the automorphism group of the induced almost Hermitian structure coincide with $\mathbf{R}^4 \rtimes SU(2)$ and it acts transitively on $S^2 \times \mathbf{R}^4$. If $\alpha = 0$ or $\pi/3$, then the automorphism group of the induced almost Hermitian structure coincides with $\mathbf{R}^4 \rtimes SO(4)$ and it also acts transitively on $S^2 \times \mathbf{R}^4$.

(4) The case of $M^6 = S^3 \times \mathbf{R}^3$, then the induced almost complex structures of $g \circ \varphi$ are different from the original that of φ. If $\alpha = 0$ or $\pi/2$, the induced almost complex structures are homogeneous. If $\alpha \in (0 < \alpha \leq \pi/2)$, the induced almost complex structures are not homogeneous.

As the theory of hypersurfaces of $\operatorname{Im} \mathfrak{C}$, we show that G_2-congruence theorem for curves of $\operatorname{Im} \mathfrak{C}$.