Results of my research

Misa Ohashi

Let $\gamma : I \to \mathbf{R}^n$ be a regular curve in an n-dimensional Euclidian space $\mathbf{R}^n$ with the canonical metric. The orientation preserving isometry group of $\mathbf{R}^n$ is the semi-direct product $\mathbf{R}^n \rtimes SO(n)$. Two regular curves $\gamma,\ \tilde{\gamma} : I \to \mathbf{R}^n$ of $\mathbf{R}^n$ are called (orientation preserving) $SO(n)$-congruent to each other if there exists $(a, g) \in \mathbf{R}^n \rtimes SO(n)$ with $\tilde{\gamma} = g \circ \gamma + a$. It is clear that if two regular curves are $SO(n)$-congruent to each other then their series of curvatures coincide, and its converse is also true (under the general condition).

In the case $n = 8$, besides the geometry of curves on $\mathbf{R}^8$ under the action of $SO(8)$ we can consider another geometry of them. A Euclidean 8-space $\mathbf{R}^8$ has a special algebraic structures, which is called the octonions $\mathbf{O}$ (or the Cayley algebra). The automorphism group of the octonions is the exceptional simple Lie group G_2 which is a Lie subgroup of $SO(7)$. There exists a faithful representation of G_2 to $End_{\mathbf{R}}(\operatorname{Im}\mathbf{O})$, where $\operatorname{Im}\mathbf{O}$ denotes the set of purely imaginary octonions. We consider the G_2 geometry $(\operatorname{Im}\mathbf{O},\ \operatorname{Im}\mathbf{O} \rtimes G_2)$, which is a subgeometry of $\mathbf{R}^7 \rtimes SO(7)$.

Two curves $\gamma,\ \tilde{\gamma} : I \to \operatorname{Im}\mathbf{O}$ with the same parameterizations and orientation are called G_2-congruent to each other if there exists $(a, h) \in \operatorname{Im}\mathbf{O} \rtimes G_2$ with

$$\tilde{\gamma} = h \circ \gamma + a.$$

The purpose of our study is the following. For a helix γ_0 in $\operatorname{Im}\mathbf{O}$, we consider the G_2-moduli space of helices

$$\{h \circ \gamma_0 : I \to \operatorname{Im}\mathbf{O} \mid h \in SO(7)\}/\sim_{G_2}.$$

Let $V_k(\mathbf{R}^7)$ be a Stiefel manifold of orthonormal k-frames in $\mathbf{R}^7$. Let S^6 and S^5 be a 6-dimensional unit sphere in $\operatorname{Im}\mathbf{O}$ and 5-dimensional unit sphere in $\mathbf{R}^6 = \{u \in \operatorname{Im}\mathbf{O} \mid \langle u, \varepsilon \rangle = 0\}$, respectively. It is well known that $S^6 \cong G_2/SU(3)$, $\quad S^5 \cong SU(3)/SU(2)$. We see that $V_2(\operatorname{Im}\mathbf{O}) \cong G_2/SU(2)$.

Although, in general for any orthonormal 3-frames $(e_1,\ e_2,\ e_3) \in V_3(\mathbf{R}^7)$, $e_1 \times e_2$ does not coincide with e_3. Therefore 3 manifolds $V_3(\mathbf{R}^7)$, $V_4(\mathbf{R}^7)$, $V_5(\mathbf{R}^7)$ can not be represented as orbits of G_2, we can observe that the curves in 3, 4, 5-dimensional Euclidian spaces of $\operatorname{Im}\mathbf{O}$ are not G_2-congruent, even if they are $SO(7)$-congruent. Therefore we need the double coset decomposition of $V_k(\mathbf{R}^7)$ $(k = 3, 4, 5)$ under the action of G_2.

We can show that

Proposition 1

$$\begin{aligned}
&\sim_{G_2} \backslash\, V_3(\boldsymbol{R}^7) = \sim_{G_2} \backslash\, (SO(7)/SO(4)) \cong [0,\ \pi]^*,\\
&\sim_{G_2} \backslash\, V_4(\boldsymbol{R}^7) = \sim_{G_2} \backslash\, (SO(7)/SO(3)) \cong \{0\} \sqcup \big((0,\ \pi) \times S^3\big) \sqcup \{\pi\},\\
&\sim_{G_2} \backslash\, V_5(\boldsymbol{R}^7) = \sim_{G_2} \backslash\, (SO(7)/SO(2)) \cong \big(\{0\} \times S^2\big) \sqcup \big((0,\ \pi) \times S^3 \times S^2\big) \sqcup \big(\{\pi\} \times S^2\big)
\end{aligned}$$

By Proposition 1, we obtain the following. Let $\mathbf{R}^k$ be k-dimensional subspace of $\operatorname{Im}\mathbf{O}$ and γ_0 be a helix in $\mathbf{R}^k$. and assume that the $(k-1)$th curvature of γ_0 is not 0. Let $\Gamma^k_{\gamma_0} = \{h \circ \gamma_0 : I \to \mathbf{R}^k \mid h \in SO(7)\}/\sim_{G_2}$ be the G_2-moduli space of helices in $\mathbf{R}^k$. Then we have

$$\begin{aligned}
&\Gamma^2_{\gamma_0} \cong \{1\},\\
&\Gamma^3_{\gamma_0} \cong \{\theta \mid \theta \in [0,\ \pi]\},\\
&\Gamma^4_{\gamma_0} \cong \{(\theta,\ \alpha,\ \sigma_{(\theta,\alpha)}(s)) \mid \theta \in [0,\ \pi],\ \alpha \in S^3,\ \sigma_\theta : I \to S^1\}.
\end{aligned}$$