Plans for the research (Satoshi Takagi)

The main goal of further researches is to apply Zariski-Rieman spaces to various topics of birational geometries, especially to those of asymptotic invariants, such as volumes of line bundles. However, there are many things to be done before we accomplish it.

For the first few years, I will concentrate on the construction of the fundamental theories of $\mathscr{A}$-schemes, and seek applications of Zariski-Riemann spaces.

a, In the sequel, we consider Zariski-Riemann spaces of profinite type. Unlike schemes, Zariski-Riemann spaces have points which do not possess affine open neighborhoods, which makes it difficult to handle them. In addition, valuation rings appear on the stalks, which are in general non-noetherian. Therefore, finitely generated modules does not behave well. However, since valuation rings are coherent rings, it is hopeful that there is no difficulty in considering coherent modules; here, a coherent module is defined as the cokernel of a morphism between locally free sheaves. In particular, any Zariski-Riemann space is a limit of birational morphisms of schemes, hence any line bundle on a Zariski-Riemann space can be described as a pull back of a line bundle on a scheme. This means that the Picard group of a Zariski-Riemann space is a colimit of Picard groups of schemes, as is pointed out in Fernex's paper [1].

In the following, we concentrate on the subject of line bundles on Zariski-Riemann spaces. In case of Zariski-Riemann spaces, we cannot consider ample line bundles, since any Zariski-Riemann space of dimension bigger than 2 cannot be embedded into a projective space, if it has exceptional loci. However, we can think of base point free line bundles, nef line bundles, and big line bundles just as for schemes, and moreover we should be able to construct a very visible theory. It is most hopeful that we can apply it to the theory of asymptotic invariants, such as volumes of line bundles.

b, I would like to finish the researches explained above within two years. After that, I will investigate the concept of differential modules and smoothness. We can define the notion of formally unramifiedness, formally smoothness and etc. using the lifting property using nilpotent extensions of commutative rings. Also, the property of finite presentation can be defined completely in a category-theoretic manner: recall that formally smoothness and locally of finite presentation implies smoothness. However, we do not know how they behave for $\mathscr{A}$-schemes. As I've already explained previously, it is important to look onto the theory of valuation rings. However, the theory of differentials does not work out on non-noetherian local rings, since it does not behave well: for example, the embedding dimension may become smaller than the Krull dimension for non-noetherian local rings.

One might hope that we can apply some method in rigid geometry to resolve this problem.

If we could pave the way through, we can hope for some applications: for example, we can visualize the relations between Cartier divisors and Weil divisors in a simpler way; we can understand the structure of singular points, or even better, we can classify them.

References

[1] S. Boucksom, T. Fernex, C. Favre: *The volume on an isolated singularity*, to appear in Duke Math. J., Preprint, arXiv: mathAG/1011.2847