Research Plan

Our main theme is infinite-dimensional quantum algebra and moduli spaces. We will treat following three topics.

AGT conjecture

AGT conjecture predicts, in terms of theoretical physics, several correspondences between four-dimensional supersymmetric gauge theory and two-dimensional conformal field theory. Among various mathematical formulations, we are mainly interested in the following one: the direct product of equivariant cohomology groups of instanton moduli spaces should have the structure of Verma module of W-algebra. The instanton moduli space has an algebro-geometric description, that is, the moduli scheme of framed torsion coherent sheaves over the projective surfaces. Moduli spaces of sheaves and W-algebras have been the central objects of our study, so this conjecture looks very interesting.

K-theoretic AGT conjecture is an analogue of this conjecture, where one considers equivariant K-groups instead of equivariant cohomology groups. The associated algebra should be the deformed W-algebra. Our purpose is to reveal the role of Ding-Iohara-Miki quantum algebra, which is a topological Hopf algebra and a relative of quantum affine algebra, behind the K-theoretic AGT conjecture.

Although various researches have been done for the original AGT conjecture, there are still a few studies for the K-theoretic version. One of the obstruction is the absence of the q-analogue of conformal field theory. We expect that Hopf algebra structure of Ding-Iohara-Miki algebra enables us to unify various viewpoints and speculations on the K-theoretic conjecture. Conversely, the geometric point of view will help us to understand the structure of quantum algebras.

Hall algebra

We have already started the study of Hall algebra, which is related to the Ding-Iohara-Miki algebra mentioned above. We proved the relation of the Hall algebra of two-periodic complexes and the Drinfeld double of the ordinary Hall algebra.

Let us explain the Hall algebra briefly. For abelian categories satisfying several conditions, Ringel introduced the Hall algebra in 1990. It is known that for the category of finite-dimensional representations of a quiver, the associated Hall algebra is the (subalgebra of) upper half of the quantum group associated to the same quiver. Recently Bridgeland introduced the Hall algebra of two-periodic complexes, which describes not the upper half but the whole part of the quantum group.

The Hall algebra of two-periodic complexes has another important property. For the abelian category of coherent sheaves over elliptic curves, Bridgeland's Hall algebra gives the upper half of Ding-Iohara-Miki algebra (due to Burban-Schiffmann and Schiffmann-Vasserot). Ding-Iohara-Miki algebra is related to the K-theoretic AGT conjecture, and one of its important property is that $\mathrm{SL}(2,\mathbb{Z})$ is included in the automorphism group. It has been expected that this part of automorphisms comes from auto-equivalences of the derived category of coherent sheaves on the elliptic curve.

We showed that under mild conditions of the abelian categories, Bridgeland's Hall algebra of two-periodic complexes coincides with the Drinfeld double of the ordinary Hall algebra. Combined with the work of Cramer, one finds that auto-equivalences of the derived category of the starting Abelian category induce automorphisms of Bridgeland's Hall algebra. In particular, we confirmed the expectation mentioned above for the elliptic curve case.

In the future work, we will discuss the relationship between several realizations and representations of Ding-Iohara(-Miki) algebra. We would like to study the relation of the Hall algebra of complexes and Bridgeland's stability conditions.

Deformed W-algebra

As for the deformed W-algebra, we would like to study the analogue of quantum Drinfeld-Sokolov reduction, which is a canonical construction of the (non-deformed) W-algebra.

The present definition of the deformed W-algebra uses free field realization, so that the study of its algebra structure and representations are ad hoc in some sense. If one has a functorial construction like Drinfeld-Sokolov reduction, one can take systematic approach for this algebra. We also expect some implication to the K-theoretic AGT conjecture.

Moreover we would like to clarify the relation between quantum groups like Ding-Iohara-Miki algebra and the deformed W-algebra. Hopf algebra technique will reveal the unknown structure of the deformed W-algebra.