2 英訳

In the previous works, we investigated and have developed the fundamental theory of schemes, so that we could establish a platform where we can treat limit spaces (such as ZR-spaces). Also there are hopes that we can apply these theories to Arakelov geometry. With these aspects in mind, we propose some themes to be worked out.

(a) **Foundation of the theory of line bundles and smoothness**

When we consider modules on ZR-spaces, we will mainly treat vector bundles; this is a natural way of thinking, since K-theory is in the limelight recently.

In the case of schemes, we can consider some properties of line bundles, such as 'ampleness', 'semiampleness', 'nefness', and 'bigness'. These definitions can be extended directly to line bundles on ZR-spaces (except for 'ampleness'). On ZR-spaces, we must treat Néron-Severi groups of infinite rank. It is worthwhile to study if we can establish similar theories like those on varieties.

On the other hand, it is also important to consider a substitute of the concept of (non-) singularity on ZR-spaces. As is pointed out by Kontsevich, smoothness of schemes can be characterized by the infinitesimal lifting property, which is also applicable to ZR-spaces. We can hope that this links to the problem of resolution of singularities in positive characteristics, which is still an open problem.

(b) **Re-establishing the foundation of Arakelov geometry**

In [4], we characterized the arithmetic compatification $\overline{\operatorname{Spec}\mathbb{Z}}$ of the Zariski spectrum of the ring of rational integers by a universal property. This can be regarded as a variant of a ZR-space. Actually, the behavior of the canonical bundle on it is more similar to nef and big line bundles than an ample bundle.

Arakelov geometry has remarkable applications, such as solving the Bogomolov conjecture; however, we must admit that the definitions of each concepts has many ad hoc part, which prevents a clear prospect of the whole theory. Now that we can characterize the arithmetic compactification via a universal property, it is very important to reexamine the whole theory in order to consider other applications to arithmetics. It is well known that the modules on $\overline{\operatorname{Spec}\mathbb{Z}}$ do not form an abelian category; however we can establish K-theories. It should be very important to describe fundamental theories such as Riemann-Roch via a common vocabulary with ordinary theories of modules on curves in algebraic geometry, which will help us to make a thorough understanding.

The two aims mentioned above are linked together, and we would like to study both simultaneously for some years.