Research Summary

[1] We consider invariant flat projective structures and flat affine structures on homogeneous spaces. It is very interesting to investigate which homogeneous space admits these geometric structures. There are many preceding results. For example it is known that an invariant flat affine structure exists on nilpotent Lie groups of 3-step (Scheuneman) or of dimension ≤ 6 (Fujiwara). These nilpotent Lie groups also admit a flat projective structure since an invariant flat affine structure induces an invariant flat projective structure. On the other hand there exist nilpotent Lie groups of dimension 10, 11 and 12 which admit no invariant flat affine structures (Benoist, Burde).

However there are so many Lie groups on which the existence problem of flat affine structures and flat projective structures have not been determined yet. In this article about the real Lie groups of dimension ≤ 5 I solved this existence problem.

[2] This article is about invariant flat complex projective structures on complex Lie groups. Simple real Lie groups admitting a flat projective structure have been already classified by Agaoka, Urakawa and Elduque, and in particular the complexification of such simple Lie groups are complex special linear groups. Furthermore before this article there were almost no examples of non-simple semisimple Lie groups admitting invariant flat projective structures. In this article we gave an infinite number of such examples.

Flat complex projective structures are described by using the atlases of manifolds. Via flat projective Cartan structures I proved the correspondence between the set of invariant flat complex projective structures on complex Lie groups and the set of infinitesimal prehomogeneous vector spaces (abbrev. PVs) over the complex field.

By using this correspondence and a classification of PVs given by Sato and Kimura I classified complex Lie groups admitting irreducible invariant flat complex projective structures. As a result about the real case we obtained semisimple real Lie groups admitting an invariant flat real projective structures, whose Lie algebras are of the form $\mathfrak{sl}(2) \oplus \mathfrak{sl}(m_1) \oplus \cdots \oplus \mathfrak{sl}(m_k)$ satisfying the equality $4 + m_1^2 + \cdots + m_k^2 - k - 4m_1m_2 \cdots m_k = 0$. In fact this equation has infinite number of positive integer solutions.

[3] About the existence problem of flat projective structures there is the following strange phenomenon: Even if two manifolds admit flat projective structures, the product manifold does not necessarily admit flat projective

structures again. The example are $SO(3)\times SO(3)$ and $S^n\times\cdots\times S^n$. Furthermore $SL(2)\times SL(2)$ does not admit any invariant flat projective structure.

In this paper from a given arbitrary projectively flat manifold M I established a construction of sequence of new projectively flat manifolds from M by using a castling transformation of prehomogeneous vector spaces. Castling transformations is an important tool for construction of prehomogeneous vector spaces and the classification. The constructed manifold has the structure of a principal fiber bundle over M and the structure group is a certain product of projective linear groups. Thus it has locally the structure of product manifolds.

Generally I constructed from a given manifold M equipped with a flat Grassmannian structure of type (n, m) an infinite sequence of projectively flat manifolds by successive castling transformations. Each projectively flat manifold N is a principal fiber bundle over M with group $\prod_{i=1}^{j} PL(k_i)$. The flat projective structure on N is described by the transition functions gluing open submanifolds of a projective space, which are same as the ones of M. Each constructed projectively flat manifolds are connected by manifolds equipped with Grassmannian structures. Moreover there is a bijective correspondence between the set of structure groups $\prod_{i=1}^{j} PL(k_i)$ of these constructed projectively flat manifolds and the set of certain solutions of a Grassmannian type quadratic equation: $\alpha\beta + k_1^2 + \cdots + k_j^2 - j - (\alpha + \beta)k_1\cdots k_j + 1 = 0$.

[4] I investigated the existence problem of invariant flar projective and affine structures from the view point of submanifolds. It is known that the borel subalgebras of complex semisimple Lie algebras admit flat complex affine structures by Y.Takemoto and S.Yamaguchi. This yields a solvable subalgebra $\mathfrak{s}$ associated to Iwasawa decomposition of a real semisimple $\mathfrak{g}$ Lie algebra also admits flat affine connection ∇. Then we can prove on the solvable subalgebras associated to Langlands decomposition of parabolic subalgebras of $\mathfrak{g}$, a flat affine connection is induced from $(\mathfrak{s}, \nabla)$.

On the other hand Y.Agaoka, H.Urakawa and A.Elduque proved that the Lie algebras of simple Lie groups admitting invariant flat projective structures are only $\mathfrak{sl}(n, \boldsymbol{R})$ and $\mathfrak{sl}(n, \boldsymbol{H})$. We showed that these Lie algebras equipped with a projectively flat affine connection induce such a connection on arbitrary parabolic sublalgebras $\mathfrak{q}$. On some parabolic subalgebras the induced affine connection ∇ on $\mathfrak{q}$ is projectively equivalent to a flat affine connection. The necessary and sufficient condition of ∇ being projectively equivalent to a flat affine connection is completely described by combinations of vertexes of the Dynkin diagram.