

Results of Applicant's Research

In the doctor course, the applicant has been interested in *superlinear elliptic equations* and has investigated properties of their solutions. Superlinear elliptic equations appear as, e.g., the standing wave solutions of the nonlinear Schrödinger equation, the stationary solutions to the mathematical model of the chemotactic aggregation and the equilibrium of stellar models. On the other hand, in mathematics, the energy functional associated with the equations often has the variational structure, and the equations are studied from a mathematical point of view.

The applicant first investigated the scalar field equation

$$\begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded domain in the N -dimensional Euclidean space \mathbf{R}^N with the smooth boundary $\partial\Omega$. The problem (1) has been studied in the case of $f(u) = O(u^p)$ as $u \rightarrow \infty$ ($p > 1$), and many results had been reported.

On the other hand, the applicant assumed that $f(u) = O(e^u)$ as $u \rightarrow \infty$ and proved that the almost similar result on the case of the power nonlinearity (here we assume $\Omega \subset \mathbf{R}^2$). Namely the energy functional corresponding to (1) has a critical point u_ϵ which attains the least positive critical value of the functional. Moreover the applicant proved that, for sufficiently small $\epsilon > 0$, there exists exactly one maximum point P_ϵ of u_ϵ . In addition it is also proved that, as $\epsilon \rightarrow 0$, P_ϵ moves to the most distant point from $\partial\Omega$.

Furthermore we investigated the profile of u_ϵ in more detail. Namely we assumed the following additional condition: there exists a unique solution w to

$$\Delta w - w + f(w) = 0 \quad \text{in } \mathbf{R}^2 \quad (2)$$

with $w(y) \rightarrow 0$ as $|y| \rightarrow \infty$. Then, under this condition, we showed that the profile of $u_\epsilon(x)$ around $x = P_\epsilon$ is close to that of $w((x - P_\epsilon)/\epsilon)$. The same result has been proved for the case of the power nonlinearity, but, in the precedent studies, the nondegeneracy of w to (2) as well as the uniqueness of w to (2) was assumed. On the other hand, the applicant proved that the nondegeneracy of w follows from the uniqueness of w and other conditions of f (in fact, for the case of the power nonlinearity, the nondegeneracy also follows from the uniqueness).

After the study above, to study the effect of the geometrical properties on the the structure of solutions to elliptic equations, the applicant began to investigate the structure of solutions to superlinear elliptic equations defined on the unit sphere, that is,

$$\begin{cases} \Delta_{\mathbf{S}^N} u + \lambda u + |u|^{p-1} u = 0 & \text{in } B_{\theta_0}, \\ u + \kappa \frac{\partial u}{\partial n} = 0 & \text{on } \partial B_{\theta_0}. \end{cases} \quad (3)$$

Here \mathbf{S}^N is the N -dimensional unit sphere ($N \geq 2$), and $B_{\theta_0} \subset \mathbf{S}^N$ is the geodesic ball whose origin is at the North Pole $(0, 0, \dots, 0, 1) \in \mathbf{R}^{N+1}$ and whose geodesic radius is $\theta_0 \in (0, \pi)$. In addition n is the outer unit normal vector at the boundary ∂B_{θ_0} .

For (3) with $N = 3$, $p = 5$, $\lambda = 0$ and $\kappa = 0$, it had been already known that there exists a positive and radial solution to (3) if and only if $\theta_0 \in (\pi/2, \pi)$ (a *radial* solution means a solution to (3) depending only on the geodesic radius from the North Pole). The applicant extended the result on $\kappa = 0$ to the case $\kappa > 0$ and clarified the structure of positive and radial solutions. Especially, by the result of the applicant, we see that a positive and radial solution to (3) exists for $\kappa > 0$ and sufficiently small $\theta_0 > 0$, which is qualitatively different from the result on $\kappa = 0$.

The result above is on positive and radial solutions. On the other hand, a result on non-positive or non-radial solutions has not been known. The applicant focus his attention on the fact and proved the existence of such solutions to (3) under $N = 2$, $1 < p < \infty$, $\kappa = 0$, $\lambda \geq 0$ and θ_0 sufficiently near π . In the study, the applicant first investigated the linearized problem of (3) around $u \equiv 0$ and proved that the multiplicity of eigenvalues for the linearized problem is 1 or 2. Next the applicant construct non-positive or non-radial solutions to (3) by using the Lyapunov–Schmidt reduction method.