Study proposal Makiko MASE

Among many examples of $K3$ surfaces, double sextic $K3$ surfaces are classically well-known and quite rich in their geometry, which is also related to the geometry of a plane curve.

Let B be a plane sextic curve with at most simple singularities. It is known that the double covering $S \to \mathbb{P}^2$ of the projective plane branching along B is a Gorenstein $K3$ surfaces so that there exists the minimal model $\widetilde{S}$ of S to be a $K3$ surface. The surface $\widetilde{S}$ obtained in this way is called a *double sextic* $K3$ *surface*. Types of singularities on the branch curves that give double sextic $K3$ surfaces are studied by Horikawa. Collect all double sextic $K3$ surfaces to form a family $\mathcal{DS}$. A double sextic $K3$ surface is identified with an anticanonical member of the weighted projective space $\mathbb{P}(1,1,1,3)$. A subfamily of $\mathcal{DS}$ is a family $\mathcal{F}_\Delta$ associated to a reflexive subpolytope Δ of the polytope

$$\Delta_{(1,1,1,3;6)} := \operatorname{Conv}\left\{(i,j,k,l) \in (\mathbb{Z}_{\geq 0})^4 \,|\, i+j+k+3l \equiv 0 \mod 6\right\}.$$

By a classification by Kreuzer and Skarke, it is a direct computation to get 3-dimensional reflexive subpolytopes of $\Delta_{(1,1,1,3;6)}$, and mirror polytope Δ^* in the sense of Batyrev for each reflexive $\Delta \subset \Delta_{(1,1,1,3;6)}$. However, there are several cases that the mirror Δ^* is no more a subpolytope of $\Delta_{(1,1,13;6)}$. Define the Picard lattice $\mathrm{Pic}(\Delta)$ associated to a reflexive polytope Δ to be the Picard lattice of the minimal model of any Δ-regular member in $\mathcal{F}_\Delta$, and $T(\Delta)$ be the orthogonal complement of $\mathrm{Pic}(\Delta)$ in the $K3$ lattice.

Problem 1 Does an isometry $\mathrm{Pic}(\Delta) \simeq T(\Delta^*) \oplus U$ hold ? Here U is the hyperbolic lattice of rank 2.

Problem 1 concerns whether or not the polytope mirror (due to Batyrev) can extend to the lattice mirror (due to Dolgachev). Moreover, following a study by Artebani-Boissière-Sarti, consider

Problem 2 Study double sextic $K3$ surfaces as 2-elementary $K3$ surface in terms of their invariants (a, r, δ), and describe the duality of a $K3$ surface with its symplectic group actions due to Nikulin.

Curves are much interesting as a subvariety of $K3$ surfaces since they are related famous mirror conjecture in a study of hypergeometric function of the period of a $K3$. In particular, in the theory of algebraic curves, the notion of Galois point (on a smooth plane curve) is introduced by Yoshihara and is well stidied. Weierstrass points are other interesting object.

For a smooth plane curve C and a point $P \in \mathbb{P}^2$, let $\pi_P : C \to \mathbb{P}^1$ be the projection of C by P. The point P is called the *Galois point* if the field extension $k(C)/k(\mathbb{P}^1)$ is Galois. For a smooth projective curve C' of genus ≥ 2, a point $P' \in C'$ is called the *Weierstrass point* if $h^0(C', \mathcal{O}(gP')) \geq 2$.

Problem 3 Study Galois/Weierstrass points of branch loci of double sextic $K3$ surfaces. Conversely, what sort of Galois/Weierstrass points should branch curves of double sextic $K3$ surfaces have ? Are they degenerate ? Are there plane sextic curves B, B' that are “dual” in some sense ? If so, is there any duality between double sextic $K3$ surfaces S_B, $S_{B'}$ branching respectively along B, B' ?

Problem 3 asks characterization of double sextic $K3$ surfaces by their subvarieties.