Research Results

Tomoyuki Nakatsuka

1. Background

My research focused on the uniqueness of solutions to the Navier-Stokes equation, which is a nonlinear partial differential equation describing the motion of a viscous incompressible fluid. In general, smallness of a solution in a suitable sense is an essential condition to guarantee the uniqueness. The coincidence of two small solutions is usually an immediate consequence of the contraction mapping theorem, however, such a uniqueness result is very restrictive. The target of my study is another type of uniqueness theorem, namely, uniqueness when one of solutions is not necessarily small. Uniqueness theorems of this type are more sophisticated. I studied the uniqueness in 3D and 2D exterior domains and obtained the results stated below. The existence of solutions which are small in a sense is well known, and my results contributed to the enlargement of the class of such solutions in which uniqueness holds.

2. 3D Exterior Problem([1], [2], [4])

The liner approximation is an important method in the analysis of nonlinear partial differential equations, and, in the L^p framework, the stationary Navier-Stokes equation requires L^3 space to estimate the nonlinearity. On the other hand, it is known that in 3D exterior domains a solution of the linearized Navier-Stokes equation, the Stokes equation, decaying faster than $|x|^{-1}$ at infinity exists only in a special situation. Thus the L^p theory does not contribute to the solvability of the nonlinear problem. We need the weak L^3 space $L^{3,\infty}$ ($\ni |x|^{-1}$) to obtain existence. The class of solutions in $L^{3,\infty}$ is important from the viewpoint of the solvability and the asymptotic behavior, and I investigated the uniqueness of solutions in this class (in the papers [1] and [2]). In particular, I proved in [2] that if one solution is small in $L^{3,\infty}$ and the other belongs to $L^3 + L^\infty$, then the uniqueness holds. The space $L^3 + L^\infty$ is critical in view of local regularity. The uniqueness under the smallness of one solution and the additional regularity of the other was proved for the first time by me. Furthermore, in [4], the joint work with Y. Taniuchi (Shinshu University) and R. Farwig (TU Darmstadt), we proved almost the same uniqueness theorem as [2] for nonstationary solutions such as time-periodic ones in $L^{3,\infty}$. This result can be regarded as the extension of my works.

3. 2D Exterior Problem([3])

The 2D exterior problem for the stationary Navier-Stokes equation is known as a challenging problem. One of the main difficulties stems from the Stokes paradox, which implies that the linear approximation is not useful in the analysis of the nonlinear problem in general. Furthermore, we cannot control the behavior of a solution with finite Dirichlet integral at infinity only from its class. Owing to these peculiar difficulties, the general theory of the existence for the 2D exterior problem is not established yet and the existence of a solution decaying at infinity is obtained only under suitable symmetry assumptions.

I investigated the uniqueness of stationary solutions with finite Dirichlet integral and some symmetry which ensures the decay of a solution. I showed that a small symmetric solution decaying like $|x|^{-1}$ is unique in the class of symmetric ones satisfying the energy inequality. Here the decay rate $|x|^{-1}$ is considered to be critical for stability. My result is the first uniqueness theorem of our type in the 2D exterior problem. As an application, we can give information on the asymptotic behavior of symmetric solutions at infinity by combining my theorem with the known result due to Yamazaki (2011).