Research plan

Geometry, representation theory, and integrable systems arising from Hessenberg varieties

Among several interesting algebraic subsets of the flag varieties, there are Springer varieties in geometric representation theory, Peterson varieties in connection with quantum cohomology rings of the flag varieties, and the toric varieties associated with root systems. The *Hessenberg varieties* provides us a unified way of describing these spaces in the flag varieties. In type A_{n-1}, they are defined from an $n \times n$ matrix and a function $h : [n] \to [n]$ satisfying certain properties.

• **Cohomology rings of Hessenberg varieties and representation of symmetric groups** [In collaboration with Mikiya Masuda and Takashi Sato and Tatsuya Horiguchi]

Motivated by the work of Brosnan-Chow, it is important to consider the family of *regular Hessenberg varieties*, and the study of cohomology rings of these Hessenberg varieties are all related to the study of cohomology rings of regular semisimple Hessenberg varieties. In this project, our aim is to provide a nice (with respect to the representation of the symmetric group) set of ring generators of the cohomology rings of the regular Hessenberg variety by using GKM theory of its torus equivariant cohomology rings.

• **Cohomology rings of nilpotent Hessenberg varieties** [In collaboration with Peter Crooks]

In this project, we study the cohomology ring of the Hessenberg variety associated to a highest root vector and a Hessenberg subspace. As special examples, we have the full flag variety and a certain Springer variety. It follows that this variety is a GKM variety. As an on-going project, we are studying some explicit presentation of its cohomology ring.

• **Newton-Okounkov bodies of Hessenberg varieites** [In collaboration with Megumi Harada and Lauren DeDieu and Jeremy Lane]

The theory of Newton-Okounkov body is a certain kind of generalization of the theory of toric varieties, and it is developing with a connection to geometry, representation theory, and combinatorics. The goal of this project is to compute the Newton-Okounkov body for Hessenberg varieites with respect to the line bundle determined by the Plucker embedding and a valuation on the field of rational functions. For example, for type A_2, we can compute a Newton-Okounkov body of the Peterson variety by using the fact that the Peterson variety can be described as a flat limit of a non-singular projective toric variety. We will also study relations between Newton-Okounkov bodies of regular semisimple Hessenberg varieites and regular nilpotent Hessenberg varieites.

• **Toric varieties associated with root systems**

As a special case of the Hessenberg varieties above, we have the toric varieity $X(\Phi)$ associated with a root system Φ. The purpose of this research project is to answer the following question: given root systems Φ_1 and Φ_1, if $X(\Phi_1)$ and $X(\Phi_2)$ are homotopic, are Φ_1 and Φ_2 isomorphic? If the root systems are irreducible of odd rank, then this is true. The case of irreducible root systems of odd rank will be the next case.