Research Plan

The applicant has been interested in and investigated the relation between the structure of solutions to superlinear elliptic equations and its domain. From now the applicant will generalize results on non-positive or non-radial solutions to

$$\begin{cases} \Delta_{\mathbb{S}^N} u + \lambda u + |u|^{p-1}u = 0 & \text{in } B_{\theta_0}, \\ u + \kappa \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{1}$$

that is, we consider (1) for $N \geq 3$. Then the linearized problem around $u \equiv 0$ is

$$\begin{cases} \Delta_{\mathbb{S}^N} w + \lambda w = 0 & \text{in } B_{\theta_0}, \\ w + \kappa \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{2}$$

and the multiplicity of eigenvalues to (2) is greater than 2 in general. Thus, the argument used in the case of $N = 2$ is not valid for $N \geq 3$, and we are required to investigate (2) with a more general point of view.

In addition the eigenvalue problem (2) contains the information of the local bifurcation structure of (1), that is, e.g., an eigenvalues of (2) is a bifurcation point of (1). Now the distribution of eigenvalues of (2) is known if θ_0 is sufficiently near π. One of our aims is to investigate eigenvalues when θ_0 is not near π.

On the other hand, we also consider the global bifurcation structure of solutions to (1). In precedent studies the case $\kappa = 0$, $\lambda = 0$ and $p > 1$ was investigated. Especially we investigated the multiplicity of positive and radial solutions for $p > (N+2)/(N-2)$. Our aim is to develop our precedent studies, and we will investigate the case $\lambda \neq 0$.

More precisely, we consider the case $-N(N-2)/4 \leq \lambda \leq \lambda_1$. Here λ_1 is the first eigenvalue of $-\Delta_{\mathbb{S}^N}$ with the homogeneous boundary condition, and the constant $-N(N-2)/4$ is related to the scalar curvature of $\mathbb{S}^N$. In addition the following equation

$$\Delta_{\mathbb{S}^N} u - \frac{N(N-2)}{4} u + |u|^{p-1}u = 0 \qquad \text{on } \mathbb{S}^N \tag{3}$$

is said to be the Yamabe equation, and there exists a positive solution to (3) (if $\lambda = 0$, then there does not exist a positive solution on the whole sphere $\mathbb{S}^N$). Furthermore, for $p = (N+2)/(N-2)$ and $\lambda < -N(N-2)/4$, it is known that there exist at least two positive solutions under some λ. On the other hand, it is also known that there exists at most one positive solution for $p = (N+2)/(N-2)$ and $-N(N-2)/4 \leq \lambda \leq \lambda_1$. Namely it seems that the structure of solutions to (1) qualitatively changes at $\lambda = -N(N-2)/4$. We focus our attention on this phenomenon and, in order to investigate that in detail, we study the structure of solutions to (1) for $-N(N-2)/4 \leq \lambda \leq \lambda_1$, especially around $\lambda = -N(N-2)/4$.

In addition we will investigate

$$\epsilon^2 \Delta u - u + u^p = 0 \qquad \text{in } \Omega$$

under the Neumann boundary condition with the critical case $p = (N+2)/(N-2)$. In this case, in general, it is not known that a positive solution exists. Hence we investigate the sufficient condition for the existence of solutions, and investigate the behavior of that as $\epsilon \to 0$.