Results of Applicant's Research

In the doctor course, the applicant has been interested in *superlinear elliptic equations* and has investigated properties of their solutions. Superlinear elliptic equations appear as, e.g., the standing wave solutions of the nonlinear Schrödinger equation, the stationary solutions to the mathematical model of the chemotactic aggregation and the equilibrium of stellar models. On the other hand, in mathematics, the energy functional associated with the equations often has the variational structure, and the equations are studied from a mathematical point of view.

The applicant first investigated the scalar field equation

$$\begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$. The problem (1) has been studied in the case of $f(u) = O(u^p)$ as $u \to \infty$ ($p > 1$), and many results had been reported. The applicant investigated the existence and the uniqueness of a positive solution to (1). Moreover he also investigated the asymptotic behavior of the positive solution as $\epsilon \searrow 0$.

After the study above, to study the effect of the geometrical properties on the the structure of solutions to elliptic equations, the applicant began to investigate the structure of solutions to superlinear elliptic equations defined on the unit sphere, that is,

$$\begin{cases} \Delta_{\mathbb{S}^N} u + \lambda u + |u|^{p-1}u = 0 & \text{in } B_{\theta_0}, \\ u + \kappa \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_{\theta_0}. \end{cases} \tag{2}$$

Here $\mathbb{S}^N$ is the N-dimensional unit sphere ($N \geq 2$), and $B_{\theta_0} \subset \mathbb{S}^N$ is the geodesic ball whose origin is at the North Pole $(0, 0, ..., 0, 1) \in \mathbb{R}^{N+1}$ and whose geodesic radius is $\theta_0 \in (0, \pi)$. In addition n is the outer unit normal vector at the boundary ∂B_{θ_0}.

For (2) with $N = 3$, $p = 5$, $\lambda = 0$ and $\kappa = 0$, it had been already known that there exists a positive and radial solution to (2) if and only if $\theta_0 \in (\pi/2, \pi)$ (a *radial* solution means a solution to (2) depending only on the geodesic radius from the North Pole). The applicant extended the result on $\kappa = 0$ to the case $\kappa > 0$ and clarified the structure of positive and radial solutions. Especially, by the result of the applicant, we see that a positive and radial solution to (2) exists for $\kappa > 0$ and sufficiently small $\theta_0 > 0$, which is qualitatively different from the result on $\kappa = 0$.

The result above is on positive and radial solutions. On the other hand, a result on non-positive or non-radial solutions has not been known. The applicant focus his attention on the fact and proved the existence of such solutions to (2) under $N = 2$, $1 < p < \infty$, $\kappa = 0$, $\lambda \geq 0$ and θ_0 sufficiently near π. In the study, the applicant first investigated the linearized problem of (2) around $u \equiv 0$ and proved that the multiplicity of eigenvalues for the linearized problem is 1 or 2. Next the applicant construct non-positive or non-radial solutions to (2) by using the Lyapunov–Schmidt reduction method.

After belonging to Osaka City University Advanced Mathematical Institute, the applicant collaborates with Yasuhito Miyamoto, an associate professor in Tokyo University, on the investigation of the structure of solutions to (2). Namely we investigated the structure of positive solutions to (2) with $\kappa = 0$, $\lambda = 0$ and $p > (N + 2)/(N - 2)$. In this case we cannot use the variational method, which is a powerful tool to prove the existence of nonlinear elliptic equations. Thus we only consider radial solutions to (2). Then (2) is reduced to ODE, and we can investigate the structure of solutions by ODE approaches.

Now, under $f(u) = u^p$ and $1 < p < (N+2)/(N-2)$, we investigate the behavior of solutions to (1) as $\epsilon \to 0$ again. In this problem we assume that the domain Ω has a cone-like boundary, that is, the boundary $\partial\Omega$ has non-regular point. In this case, if $\epsilon \to 0$, then the least energy solution concentrates at the sharpest non-regular point. I prepare to present this result now.