# Plan of Research

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The work of Alday-Gaiotto-Tachikawa in 2009 leads to a renewed attention to the connection between two-dimensional conformal field theories and four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The two-dimensional correlation function (conformal block) is identified with the instanton partition function of the gauge theory.

This "2d/4d connection" has been generalized to various directions. One of these generalizations is a correspondence between the "$q$-deformed" two-dimensional conformal field theories and the "q-lifted" five-dimensional supersymmetric gauge theories. The $q$-deformed 2d theories have $q$-Virasoro or $q$-W algebras, while for the $q$-lifted 5d theories, the 5-th direction is compactified on a circle $S^1$ whose radius is proportional to $\log q$.

In the 2d theory, the $q$-Virasoro algebra plays a central role. Its representation theory is still not well understood although 20 years have passed since its introduction. In particular, for a $q$-deformation of primary fields in the conformal field theory, the first principle definition of $q$-vertex operators is not known.

In our recent work, based on the "2d/5d connection", we have obtained an operator realization of $q$-block which coincides with the Nektrasov partition function of the five-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory. This means that a $q$-vertex operator is explicitly determined from the 5d Nekrasov function. Hence we also obtained a Coulomb gas representation (Dotsenko-Fateev integral representation) of the $q$-deformed block. We will study various properties of this $q$-primary operator ($q$-vertex operator).

There is a subtle point in the $q$-deformed Coulomb gas representation. If one uses one kind of vertex operators, a insertion point of one of $q$-vertex operators must be slightly shifted from a natural position in order to match with the Nekrasov function. At this moment, the reason why this shift is necessary is unknown. Thus, we will also consider this point.

In our work, we showed that by taking the root of unity limit of $q$, the parafermion algebra can be obtained from the $q$-Virasoro or $q$-W algebras. We will also study this limit in more detail, and try to elucidate various properties of the 2d/4d connection. One of future theme of our study is the extension to the gauge theories on general ALE spaces or to quiver gauge theories. It is proposed an construction of associated "Yangian" for the quiver gauge theories. In the Schur-Weyl duality, the Yangian is connected with a Hecke algebra. Hence, an extension to these direction will deepen our understanding of the gauge/CFT/matrix model correspondence.

The "2d/5d connection" leads to various "2d/4d connections". Similary, this "2d/5d connection" may be obtained from more fundamental connection. We will try to understand these connections from wider perspective.