Plan of the future studies

(1) Virasoro algebra and the elliptic Calogero-Sutherland model

It is known that the Calogero-Sutherland model has the symmetry of the Virasoro algebra. Further there exists an elliptic analog of the Calogero-Sutherland model, called the elliptic Calogero-Sutherland model. Thus we may guess that the elliptic Calogero-Sutherland model has connections to the free field realization of the Virasoro algebra on the torus. On the other hand, Langmann have studied the elliptic Calogero-Sutherland model by the free field realization. Therefore it would be possible to reformulate some results on the elliptic Calogero-Sutherland model based on the viewpoint of the free field realization of the Virasoro algebra on the torus.

(2) An elliptic analog of the Macdonald polynomials

Since the trigonometric degeneration of the Ruijsenaars model can be solved due to the theory of the Macdonald polynomials, the studies of the Ruijsenaars model are directly connected with the studies of an elliptic analog of the Macdonald polynomials. Langmann has derived the functional equations of the kernel function for the elliptic Calogero-Sutherland Hamiltonian. From the functional equation, we can obtain the solutions to the elliptic Calogero-Sutherland model with p-derivative, where p is the elliptic modulus. On the other hand, the author have derived the functional equation of the kernel function for the Ruijsenaars operator by the free field realization. Then we notice that our functional equation is a q-analog of Langmann's one. Thus the functional equation of the kernel function for the Ruijsenaars operator contains important informations of an elliptic analog of the Macdonald polynomials.

(3) Modular double of the Ding-Iohara-Miki algebra

Modular doubles of quantum groups are quantum algebras which have some modular properties. For example, modular double of $U_q(sl(2,\mathbb{R}))$ written by

$$U_{q\widetilde{q}^{-1}}(sl(2,\mathbb{R})) = U_q(sl(2,\mathbb{R})) \otimes U_{\widetilde{q}^{-1}}(sl(2,\mathbb{R}))$$

consists of a pair of quantum groups commuting with each other. Here parameters $q \in \mathbb{C}^\times$ and its modular transformed parameter $\widetilde{q} \in \mathbb{C}^\times$ are defined by $q = e^{i\pi b^2}$, $\widetilde{q} = e^{-i\pi b^{-2}}$ ($b \in \mathbb{R} \setminus \mathbb{Q}$). Thus the parameters q, $\widetilde{q}$ are on the unit circle $|q| = |\widetilde{q}| = 1$. It is known that the universal R operator of $U_{q\widetilde{q}^{-1}}(sl(2,\mathbb{R}))$ is written by the double sine function.

Let us consider whether there exists the modular double of the Ding-Iohara-Miki algebra. The author have checked that from a kernel function defined by the double sine function, the free field realization of the modular double of the Ding-Iohara-Miki algebra arises. On the other hand, we notice that the kernel function written by the double sine function is a scaling limit of the kernel function for the elliptic Ruijsenaars operator. This means that representations of the modular double of the Ding-Iohara-Miki algebra would have connections to those of the **elliptic** Ding-Iohara-Miki algebra.