We have considered the following special case :

Question Can any transpose-dual pair $(B,\,B')$ (in the sense of Ebeling-Ploog) extend to a lattice dual pair, namely, when the singularity B (resp. B') is associated to a family $\mathcal{F}_\Delta$ (resp. $\mathcal{F}_{\Delta'}$) of $K3$ surfaces, does the duality

$$(\sharp) \qquad \mathrm{Pic}(\Delta)^{\perp}_{\Lambda_{K3}} \simeq U \oplus \mathrm{Pic}(\Delta')$$

hold ?

In [2], we found some examples of transpose-dual pairs of singularities that extend to the duality $(\sharp)$. However, there were still open cases existed.

So, in [1], we have found candidates for satisfying the duality $(\sharp)$ by constructing appropriate families :

Main Theorem. *For each of the following transpose-dual pairs*

$$(Z_{1,0},\,Z_{1,0}),\,(U_{1,0},\,U_{1,0}),\,(Q_{17},\,Z_{2,0}),\,(W_{1,0},\,W_{1,0}),$$

there exist compactifications F, F' of defining polynomials and reflexive polytopes Δ and Δ' such that the conditions $(**)$ are satisfied:

$$(**) \qquad \Delta^* \simeq \Delta',\,\Delta_F \subset \Delta,\,\Delta_{F'} \subset \Delta', \quad \textit{and} \quad \mathrm{rk}\,L_0(\Delta) = 0.$$

Therefore, the pair of associated families $\mathcal{F}_\Delta$ and $\mathcal{F}_{\Delta'}$ is a canditate of the duality $(\sharp)$ in the sense that for the ranks $\rho(\Delta)$, $\rho(\Delta')$ of Picard lattices of the families, $\rho(\Delta)+\rho(\Delta')=20$ holds. □

In [1], it is conjectured that there do not exist reflexive polytopes for transpose-dual pairs $(Z_{13},\,J_{3,0})$, $(Z_{17},\,Q_{2,0})$, $(U_{16},\,U_{16})$, $(W_{17},\,S_{1,0})$, $(W_{18},\,W_{18})$, $(S_{17},\,X_{2,0})$ of singularities satisfying the condition $(**)$.

In conclusion, we also conjecture that for each of the dual pairs

$$(Z_{1,0},\,Z_{1,0}),\,(U_{1,0},\,U_{1,0}),\,(Q_{17},\,Z_{2,0}),\,(W_{1,0},\,W_{1,0}),$$

the pair of families $(\mathcal{F}_\Delta,\,\mathcal{F}'_{\Delta'})$ obtained in the **Main Theorem** satisfies the relation

$$(\sharp) \qquad \mathrm{Pic}(\Delta)^{\perp}_{\Lambda_{K3}} \simeq U \oplus \mathrm{Pic}(\Delta').$$

References

[1] Mase, M. Polytope duality for families of $K3$ surfaces associated to transpose duality. *to appear in Commentalli Math. St. Pauli.*.

[2] Mase, M. A mirror duality for families of $K3$ surfaces associated to bimodular singularities. *Manuscripta Math.(online 26 September 2015) 149*, 3–4 (2016), 389–404.