We define the Picard lattice of a family of $K3$ surfaces as hypersurfaces in a Fano 3-fold (we call it simply a family of $K3$ surfaces) to be the Picard lattice of $K3$ surfaces as the minimal model of any generic section of the family of $K3$ surfaces. A toric threefold that is constructed from a 3-dimensional reflexive polytope is a Fano 3-fold.

(a) Birational equivalence of families of $K3$ surfaces

We studied whether or not general sections in families of $K3$ surfaces are birationally equivalent if the Picard lattices of the families are isometric for the cases when the Fano threefolds are weighted projective spaces in [Kobayashi-Mase, 2012], and for certain non-singular Fano 3-folds in [Mase, 2012] [Mase, 2014].

(b) Duality of singularities and $K3$ surfaces

[Ebeling-Takahashi][Ebeling-Ploog] showed that there exists a "transpose duality" for certain isolated hypersurface singularities in $\mathbb{C}^2$ which were classified by [Arnold].

Theorem(Sammary of the results) [Mase-Ueda, 2015, Mase, 2016–17]

(1) For any transpose-dual pairs, there exist 3-dimensional reflexive polytopes Δ, Δ' that are polar dual to each other, and deformations F, F' of defining polynomials of singularities f, f', such that the Newton polytope of F (resp. F') is a subpolytope of Δ (resp. Δ').

(2) For any transpose-dual pairs in Table 1, the natural restriction of the $(1,1)$-Hodge component from that of the projective space associated to Δ to that of generic member of the family of $K3$ surfaces is surjective.

Moreover, the Picard lattices of the families of $K3$ surfaces associated to Δ and Δ' are given as in Table 1, and the lattices Pic_Δ and $U \oplus \mathrm{Pic}_{\Delta'}$ are orthogonal to each other in the $K3$ lattice $\Lambda_{K3} := U^3 \oplus E_8^2$. In the following, the names of singularities follow [Arnold], and the names of lattices [Bourbaki]. The rank is denoted by $\rho_\Delta := \mathrm{rk}\,\mathrm{Pic}_\Delta$. The discriminant of a lattice K is denoted by $\mathrm{discr}\,K$. In particular, lattices K_1, K_2 are negative-definite and even with invariants being $\mathrm{rk}\,K_1 = 15$, $\mathrm{discr}\,K_1 = -18$, and $\mathrm{rk}\,K_2 = 16$, $\mathrm{discr}\,K_2 = 4$.

B	Pic_Δ	ρ_Δ	$\lvert\mathrm{discr}\rvert$	$\rho_{\Delta'}$	$\mathrm{Pic}_{\Delta'}$	B'
Q_{12}	$U \oplus E_6 \oplus E_8$	16	3	4	$U \oplus A_2$	E_{18}
$Z_{1,0}$	$U \oplus E_7 \oplus E_8$	17	2	3	$U \oplus A_1$	E_{19}
E_{20}	$U \oplus E_8^{\oplus 2}$	18	1	2	U	E_{20}
$Q_{2,0}$	$U \oplus A_6 \oplus E_8$	16	7	4	$U \oplus \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$	Z_{17}
E_{25}	$U \oplus E_7 \oplus E_8$	17	2	3	$U \oplus A_1$	Z_{19}
Q_{18}	$U \oplus E_6 \oplus E_8$	16	3	4	$U \oplus A_2$	E_{30}
$Z_{1,0}$	$U \oplus D_5 \oplus E_7$	14	8	6	$U \oplus A_1 \oplus A_3$	$Z_{1,0}$
$U_{1,0}$	$U \oplus K_1$	17	18	3	$\begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} \oplus A_1$	$U_{1,0}$
Q_{17}	$U \oplus E_6 \oplus E_7$	15	6	5	$U \oplus A_1 \oplus A_2$	$Z_{2,0}$
$W_{1,0}$	$U \oplus K_2$	18	4	2	$\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$	$W_{1,0}$

Table 1: *Picard lattices of families of $K3$ surfaces that are of lattice-dual*

(c) Classification of singular fibres of Jacobian elliptic singular $K3$ surface with transcendental lattice being $(2) \oplus (6)$

The classification is equivalent to that of primitive embeddings of a "trivial lattice" $M = A_1 \oplus A_5$ into the Neimeier lattice of rank 24. We use this fact to achieve the result. [BERTIN, GARBAGNATI, HORTSCH, LECACHEUX, MASE, SALGADO, and WHITCHER, 2015]