Research plan

Geometry, representation theory, and integrable systems arising from Hessenberg varieties

Among several interesting algebraic subsets of the flag variety, there are Springer varieties in geometric representation theory, Peterson varieties in connection with quantum cohomology rings of the flag varieties, and the toric varieties associated with root systems. The *Hessenberg varieties* provides us a unified description of these spaces in the flag variety. In type A_{n-1}, they are defined from an $n \times n$ matrix X and a function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying certain properties as follows:

$$\mathrm{Hess}(X, h) = \{V_\bullet \in Fl(\mathbb{C}^n) \mid XV_j \subseteq V_{h(j)} \ (j = 1, \ldots, n)\}.$$

In general, let G be a complex semisimple algebraic group, $B \subset G$ a Borel subgroup, $\mathfrak{g}$ and $\mathfrak{b}$ the Lie algebras of G and B, respectively. For an element $x \in \mathfrak{g}$ and a B-invariant linear subspace $\mathfrak{b} \subseteq H \subseteq \mathfrak{g}$, the Hessenberg variety associated with x and H is defined to be

$$\mathrm{Hess}(x, H) = \{gB \in G/B \mid \mathrm{Ad}(g^{-1})x \in H\}.$$

The following are my research plan.

• **Weyl type character formula for Hessenberg varieties** [In collaboration with Haozhi Zeng and Naoki Fujita]

As special cases of regular semisimple Hessenberg varieties in type A, we have the flag variety and the toric variety associated with the fan consisting of the Weyl chambers. The flag variety is Fano, and this toric variety is weak Fano. These facts led us to study when a given regular semisimple Hessenberg variety is weak Fano. (It is easy to characterize when it is Fano.) For such a Hessenberg variety, any torus equivariant ample line bundle has trivial higher cohomology groups, and the space of global sections admits a Weyl type character formula. This is a first step to the problem whether there is toric degenerations and a completely integrable system on regular semisimple Hessenberg varieties.

• **The singular loci of regular nilpotent Hessenberg varieties** [In collaboration with Erik Insko]

A regular nilpotent Hessenberg variety in type A is a singular variety except for the flag variety, and we study how to describe its singular locus. For the fixed points of a natural $\mathbb{C}^*$-action, it is easy to characterize when it is a singular point in terms of combinatorial information of h. So, we investigate how to generalize this to arbitrary given point. It is also interesting to study singular loci of a (possibly non-nilpotent) regular Hessenberg varieties.