Research plan

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1. Solutions of the dynamical systems of Buchstaber and Mikhailov for any genus

Buchstaber and Mikhailov introduced the polynomial dynamical systems on $\mathbb{C}^4$ on the basis of commuting vector fields on the symmetric square of hyperelliptic curves. For genus 1, the systems are integrated in terms of the Weierstrass ζ functions. For genus 2, the systems are integrated in terms of the hyperelliptic functions of genus 2 [1]. We constructed the solutions of the systems for genus 3 in terms of the meromorphic functions on the sigma divisor. We will construct solutions of the systems for any genus. In order to do that, it is necessary to express the coordinates of 2 points on a hyperelliptic curve of genus g from their Abel-Jacobi image. For $g-2 \leq k \leq g$, the formulae that express the coordinates of k points on the hyperelliptic curve of genus g from their Abel-Jacobi image are known. We will extend the formulae for $k < g-2$.

2. Construction of the deformation of the KdV-hierarchy for any genus

The Weierstrass elliptic function $\wp$ satisfies the equation $(\wp')^2 = 4\wp^3 - g_4\wp - g_6$, where $g_4, g_6 \in \mathbb{C}$. For the hyperelliptic curves of genera 2 and 3, Baker obtained explicit expressions for the higher logarithmic derivatives of sigma functions of many variables in the form of polynomials in logarithmic derivatives of the second and the third order of these functions. Buchstaber, Enolskii, and Leykin extended the Baker's formulae to hyperelliptic curves of any genus. These formulae show that the hyperelliptic functions satisfy the KdV-hierarchy for any genus. Let $\sigma^{(g)}$ be the hyperelliptic sigma function of genus g and $W_g = \{w \in \mathbb{C}^g \mid \sigma^{(g)}(w) = 0\}$. For $g = 3$, we constructed the two parametric deformation of the KdV-hierarchy by using the dynamical systems of Buchstaber and Mikhailov. Further, we constructed the solutions of the new system in terms of the meromorphic functions on the sigma divisor by using the Abel-Jacobi map. The new system can be also derived from the relations of the sigma functions $\sigma^{(3)}$ on W_3, which are derived from the Baker's formulae. For any g, we will derive the relation of the sigma functions $\sigma^{(g)}$ on W_g from the relations of the hyperelliptic functions and construct the deformation of the KdV-hierarchy.

3. Series expansion of the sigma function of arbitrary Riemann surface

The coefficients of series expansion of the Riemann theta function are given by matrices of periods. On the other hand, the coefficients of series expansion of the sigma function around the origin are polynomials of the coefficients of the defining equations of algebraic curves with rational numbers. This fact, which plays an important role in applications, distinguishes the sigma function from the Riemann theta function. The sigma function is extended to any Riemann surface [3]. A model of algebraic curves, which expresses all algebraic curves, is known (Miura canonical form) [2]. We will express the defining equations of arbitrary algebraic curve by Miura canonical form and show that the coefficients of the series expansion of the sigma funciton of arbitrary Riemann surface [3] are polynomials of the coefficients of the defining equations with rational numbers.

References

[1] V. M. Buchstaber, A. V. Mikhailov, "The space of symmetric squares of hyperelliptic curves and integrable Hamiltonian polynomial systems on $\mathbb{R}^4$", arXiv:1710.00866, (2017).

[2] S. Miura, "Linear Codes on Affine Algebraic Curves", *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, Vol. J81-A, Issue 10, (1998), 1398–1421.

[3] A. Nakayashiki, "Tau Function Approach to Theta Functions", *International Mathematics Research Notices*, Volume 2016, Issue 17, (2015), 5202–5248.