Abstract of future research

For the details of some notations, refer to abstract of present research.

1. **Error estimate of approximation**: At this stage, we don't know that the estimate

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \leq CT^{1/4}(a_n)E_{p,n}(w;f). \tag{1}$$

   is sharp or not. We may decrease more the power of T by technique of proof. And we also may weaken the condition of an Erdős-type weight, that is $T(a_n) \leq c\,(n/a_n)^{2/3}$.

2. **Estimates of derivatives of the de la Vallée Poussin mean**: We also show L^p boundedness of derivatives of the de la Vallée Poussin mean. One of these is the following: Suppose that w belongs to $\mathcal{F}_\lambda(C^4+)$ which is a smooth subclass of $\mathcal{F}(C^2+)$. If $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then for $2 \leq p \leq \infty$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^p(\mathbb{R})} \tag{2}$$

   and for $1 \leq p \leq 2$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^j a_n^{(2-p)/2p}\|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}$$

   for all $1 \leq j \leq k$ and $n \in \mathbb{N}$. We use duality of L^1-norm and Riesz-Thorin interpolation theorem to prove L^p boundedness of the de la Vallée Poussin mean. But, unfortunately, we cannot use duality of L^1-norm because T remains in the proof and it is unbounded. So we could know (2) holds true or not for $1 \leq p \leq 2$. We would like to find the way to break through obstructions by unboundedness of T.

3. **Weighted approximation by Lagrange interpolation polynomials**: We are going to study the Lagrange interpolation polynomials $L_n(f)$ for $w^* \in \mathcal{F}(C^2+)$. Here, f is a continuous function on $\mathbb{R}$. We need to show

$$\lim_{n\to\infty} \|(f - L_n(f))w\|_{L^p(\mathbb{R})} = 0.$$

   So far, when we prove some estimate for Erdős-type weight, we use mollification of the weight for $w \in \mathcal{F}_\lambda(C^3+)$. If we use this, the weight $T^\alpha w$ changes a new weight $w^* \in \mathcal{F}(C^2+)$. Because $L_n(f)$ depends on orthogonal polynomials $\{p_n\}$ with respect to w, but also the zeros of p_n, It is difficult to use mollification of the weight. One of our problem is to find a way which to use mollification of the weight as less as possible.

4. **Laguerre-type weights**: In addition, as an application of above subject, we will study the case of $\mathbb{R}^+ := [0, \infty)$. This study have a connection with the theory of Laguerre polynomials. First, we are going to define a relevant class of weights on $\mathbb{R}^+$ in response to $\mathcal{F}(C^2+)$ on $\mathbb{R}$ and show property of its orthogonal polynomials, MRS number and the function correspond to T of $\mathcal{F}(C^2+)$.