Plan of Future Research

Basically, I continue and develop the research carried out until now, particularly for around these 20 years, consistently done in various ways for algebra, geometry and analysis on homogeneous open convex cones and on complex or real homogeneous Siegel domains. Since the research so far has satisfactorily achieved results, I think that there is no need to drastically change the direction and the method of research. In particular, I would like to continue working on analysis on domains gotten by representations of Jordan algebras as treated in the joint work with Nakashima. I would also like to attack more detailed algebraic structures of homogeneous open convex cones with the active use of oriented graphs which I have exploited in the joint work with Yamasaki. In what follows, I describe short-term plans 1), 2), and a long-term plan 3).

1) **Improvement of basic theory of homogeneous open convex cones**. Foundation of basic theory for homogeneous convex cones is laid by Vinberg's paper published in 1963. However, that paper has a big gap in proving the convexity of a homogeneous open cone defined by a non-associative algebra. This gap has been closed by a geometric argument developed by Koszul as described in Chapter 8 of the book "H. Shima, Geometry of Hessian Structures", and also by an analytic argument given by a paper of H. Rossi and M. Vergne published in 1973. I found an algebraic argument by making more precise the original Vinberg's argument (though the inductive argument involves dual cones and becomes large-scaled), and a Master course student Satoshi Tanaka completed the proof in his Master Thesis submitted to Kyushu University in 2017. In order to turn this master thesis to an academic paper, I feel that some brute-force discussions and computations need to be sophisticated. I would like to settle this sophistication as an immediate first research achievement objective and to write up an academic paper.

2) **Minimum size matrix realization of homogeneous open convex cones**. In the conference "Geometry, Representation Theory, and Differential Equation" held at Kyushu University in February 2016, I gave a talk entitled "Optimal matrix realization of homogeneous cones", and I would like to write up an academic paper which details the contents of that talk as a second immediate research achievement next to the above 1). This matrix realization describes any given homogeneous open convex cone Ω as a slice $Z \cap \mathscr{P}(N, \mathbb{R})$ of the homogeneous convex cone $\mathscr{P}(N, \mathbb{R})$ of positive definite matrices in the vector space $\mathrm{Sym}(N, \mathbb{R})$ of real $N \times N$ symmetric matrices with a subspace Z of $\mathrm{Sym}(N, \mathbb{R})$ by taking an appropriate positive integer N. There are many such realizations as described by a paper of Graczyk–Ishi published in 2014, but I give a minimum N_0 of N by applying the research result 4) with Yamasaki. The formula for this N_0 is expressed concretely by using the terms and the informations of the oriented graph drawn with the algebraic structure of Ω.

3) **Analysis on homogeneous open convex cones with good properties**. I have been continuing the research of homogeneous open convex cones and homogeneous Siegel domains from the viewpoint that they are higher rank case of Damek–Ricci spaces which are of rank one. The Damek–Ricci space is a non-symmetric non-compact harmonic space, and as such the Damek–Ricci spaces attracted a great geometric interest, too. In addition to the above, the Damek–Ricci spaces suggested a new development of analysis on solvable homogeneous spaces. Symmetric spaces are already well-selected mathematical objects and there are many suitable examples even in

higher rank cases, and through experiments in such examples, general theory has been constructed. Turning to the non-symmetric cases, we did not have enough higher rank good examples with rich structures in the past, nor enough researches which extract a series of such concrete examples. However, by my research of these 15 years, such unsatisfactory status is being remedied gradually, so that on the basis of my research achievements, I would like to develop analysis on sufficiently well-selected homogeneous convex domains, on real or complex homogeneous Siegel domains. I found a way of making use of oriented graphs, and this might strengthen my tools for researches. The plan contains the research of non-symmetric domains defined by starting with symmetric domains, so that my research would clarify various already established facts about symmetric domains from a new angle at the same time.