（英訳文）

Preface

I major in complex geometry and related topics. Especially, I am interested in the problem of finding a "canonical geometric structure" in a given (co)homology class on manifolds. I have dealt with the problem by using mainly geometric analysis methods. Concretely, my reserch interests are

- compact complex manifolds X and holomorphic line bundles $L \to X$
- oriented compact symplectic 4-manifolds M, and its 2-dimensional submanifold Σ

Here the former case, we try to find a canonical Kähler metric in the space of Kähler metrics $\mathcal{H}_L$ contained in the first Chern class $c_1(L)$. Usually a "canonical metric" means a metric with constant curvature, and hence the former problem can be thought as a generalization of the classical uniformization theorem for compact Riemann surfaces. On the other hand, in the later problem, we assume that the manifold M is symplectic, and consider *when the manifold can be deformed to a Kähler one.* Where the (co)hololology class under consideration is determined by the symplectic form, or as a cycle generated by Σ. A difference to the former case is that we can consider some typical geometric structure (anti-self dual metric, hyperkähler, etc.) due to the restriction of the dimension. While these two areas are not usually put side-by-side, both problem originates from some variational principle for energy functionals like harmonic forms, and hence it is worth discussing them together. In particular, **geometric flows** obtained by the gradient flows of energy functionals play an important role to construct a canonical geometric structure. This is just a problem of solving elliptic/parabolic equations in nature, while by using the discretization method, one can deal with it from the view point of **dynamical systems**. Moreover, since X is projetive, one can ask the equivalence between the existence of canonical metrics and algebro-geometric stabilities (**GIT stability**). Thus my reserch lies at the intersection of geometry, analysis and algebra, based on the rich geometric structures. In what follows, we will take a close look at each theme.

Research (1) Geometric flows

We say that X is Fano if the line bundle L is isomorphic to the anti-canonical line bundle $-K_X$. Then the **Kähler-Einstein (KE) condition** $\mathrm{Ric}(g) = g$ is one of the most natural curvature condition for Kähler metrics, and the Kähler-Ricci flow (KRF) is a very famous evolution equation designed to deform any Kähler metrics to the KE one. The KRF is studied by many experts and there are large amount of works so far, while little is known for general L, or non-compact manifolds. For this reason, in [Result 2], we made some attempt to extend the KRF for general polarization L. In order to study canonical metrics, it is standard to consider manifolds with large symmetries. In particular, projective bundles on manifolds with constant scalar curvature, called the **Calabi ansatz** is a candidate for such spaces. On the Calabi ansatz, one can reduced the equation of canonical metrics to an ODE due to the symmetry. As a result, I discovered a necessary and sufficient condition for X admitting a constant scalar curvature Kähler metric, and moreover, I extended the KRF for general polarizations and proved the exponential convergence of the flow under some appropriate assumption. Next, in [Result 7], I studied the evolution of KRF having a conial singularity along a divisor $D(\subset X)$. This can be thought as an extension of KRF to the non-compact manifolds $X\backslash D$. From the algebro-geometric view point, it is important to understand the pair of manifolds and divisors (X, D). Consequently, I can construct the long-time solution as the limit of a family of (smooth) KRF's. Moreover, in [Result 8], we introduced a new parabolic flow, called the **inverse Monge-Ampère flow** (MA^{-1}-flow), and studied the long-time existence and convergence. A motivation for studying the MA^{-1}-flow is that it has interesting behaviors when X does not admits any KE metrics (see Reserch (3) for details).

Now, I recall a fundamental (but important) question for submanifolds proposed by Tian and Yau in the late 1990s, "*Can a given symplectic submanifold Σ be deformed to a holomorphic one in Calabi-Yau (Ricci flat Kähler) surfaces X?*". Holomorphic curves in Calabi-Yau surfaces are important in the study of superstring theory or mirror symmetry in Physics, and have catched a great deal of attention from many mathematician and physicists. In [Result 9], we reviewed this problem from the view point of hyperkähler geometry of M. For any 2-dimensional closed submanifold Σ, we introduced some energy concept, called the "**twistor energy**" by using the structure of twistor spaces, and as a consequence, we showed that any Σ with sufficiently small twistor energy can be deformed isotopically to a holomorphic one along the mean curvature flow. This result indicates that any holomorphic curve is stable along the mean curvature flow, as well as the twistor energy causes some gap for any hyperkähler manifold M which admits no holomorphic curves (Gap theorem of twistor energies).

Research (2) Dynamical Systems

The geometric quantization[*4] is a dynamical system which approximates the space of Kähler metric $\mathcal{H}_L$ by means of the space of Fubini-Study metrics $\mathcal{H}_k$ on $\mathbb{P}(H^0(L^k))$ via the projective embeddings $X \hookrightarrow \mathbb{P}(H^0(L^k))$ by letting $k \to \infty$:

$$\mathcal{H}_L = \overline{\bigcup_{k \geq 1} \mathcal{H}_k}.$$

The above formula was proved by Bouche-Catlin-Tian-Zeldich, where the closure in the RHS is taken w.r.t. the C^∞-topology on X. In the quantization process, we can regard the solution to a PDE on $\mathcal{H}_L$ as a limit of an algebraic equation on $\mathcal{H}_k$ when $k \to \infty$ in some sence. The geometric quantization is a kind of dynamical systems, specific to projective manifolds, and studied by many experts since it has a deep connection with the asymptotic expansion of Bergman kernels.

In [Result 1], I construct a quantization scheme of KE metrics. Then as a parabolic analogy of this, I also constructed a quantization for the KRF in [Result 5]. Both of these proofs are based on a variation principle, i.e. KE metrics/KRF can be characterized as the critical point/gradient flow of some energy functional on $\mathcal{H}_L$ respectively, so it is natural to study the convexity and valuation of it. Especially, the Hessian of energy functional has a connection with the asymptotic expansion of Berezin-Toepliz operators, and I studied some properties of it in [Result 4]. On the other hand, in [Result 10], we introduce a new dynamical system including the Ricci operator to construct coupled Kähler-Einstein metrics[*5].

Research (3) GIT stability

Roughly speaking, the space of Kähler metrics $\mathcal{H}$ has a good shape when X admits a canonical metric, and then the behavior of geometric flow characterizes the geometry of $\mathcal{H}$. On the other hand, in the study of GIT (Geometric Invariant Theory) stabilities, we study the structure of the boundary $\partial\mathcal{H}$ by using more alogeblo-geometric methods, and ask if there is some relation between the geometry of $\partial\mathcal{H}$ and that of $\mathcal{H}$. Concretely, our aim is to formulate a GIT stability in terms of deformation families of polarized manifolds (X, L) to schemes, called the **test configurations**, study the equivalence to the existence of canonical metrics, and exploit formulas to check the stability. Actually, in [Result 6], we formulated a GIT stability corresponding to the quantized KE metrics constructed in [Result 1], and studied the relation to the Chow stability[*6]. On the other hand, in [Result 3], we exploit a formula for checking the GIT stability for a "generalized KE metrics" (including a hamiltonian of a vector field) in terms of projective data (like homogeneous polynomials) via projective embeddings.

Meanwhile, little is known about *how to study Fano manifolds admitting no KE metrics*. In this case, there is at least one test configuration which destabilizes the Fano manifold X. In particular, we are interested in the one which optimally destabilizes the Kähler/algeblo-geometric structure of X is some suitable sense. In [Result 8], we studied the case when X is toric, and showed that the MA^{-1}-flow we introduced indeed encodes the optimal destabilizer as a solution to some optimization problem of a energy functional on $\partial\mathcal{H}$ as well as its algebraic structure. In particular, the optimal destabilizer is canonically determined from X,

[*4] The name "geometric quantization" comes from quantum mechanics in physics, and the parameter $1/k$ corresponds to the "Planck constant".

[*5] A generalization of KE metrics drawing its motivation from the "N-body problem" in Physics.

[*6] Chow stability is a kind of GIT stabilities

and can be regarded as a generalization of KE metrics in this sence. As far as I know, such a phenomena is not found in the study of KRF, and hence makes a difference between the MA^{-1}-flow and KRF.