## Abstract of present research


Our study is polynomial approximation theory as an application of potential theory. On $\mathbb{R}$, every polynomial $P(x)$ blows up as $|x| \to \infty$, we must multiply a weight function $w(x)$. Then, for $1 \le p \le \infty$ and $fw \in L^p(\mathbb{R})$, is there exist a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|(f - P_n)w\|_{L^p(\mathbb{R})} = 0 \qquad \text{(A)}$$

holds? We assume that an exponential weight $w$ belongs to relevant class $\mathcal{F}(C^2+)$. Let $w$ be $w(x) = \exp(-Q(x))$. We consider a function $T(x) := xQ'(x)/Q(x),\ (x \neq 0)$. If $T$ is bounded, then $w$ is called a Freud-type weight, and otherwise, $w$ is called an Erdős-type weight. In this study, we consider Erdős-type weights.

1. **Convergence of the de la Vallée Poussin mean**: The de la Vallée Poussin mean $v_n(f)$ of $f$ is defined by $v_n(f)(x) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(x)$, where $s_m(f)(x)$ is the partial sum of Fourier series of $f$ for orthogonal polynomials with respect to $w$. The degree of approximation for $f$ defined by $E_{p,n}(w; f) := \inf_{P\in\mathcal{P}_n} \|(f - P)w\|_{L^p(\mathbb{R})}$. Here, $\mathcal{P}_n$ is the set of all polynomials of degree at most $n$. We assume that $w \in \mathcal{F}(C^2+)$ and suppose that $T(a_n) \le c\,(n/a_n)^{2/3}$ for some $c > 0$. Here, the notation $a_n$ is called MRS number. Then there exists a constant $C \ge 1$ such that for every $n \in \mathbb{N}$ and when $fw \in L^p(\mathbb{R})$,

   $$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; f). \qquad \text{(B)}$$

   We show the following two conditions such that the right side of (B) converge to 0 as $n \to \infty$. First, if $w$ belongs to smooth subclass $\mathcal{F}_\lambda(C^3+)$ and $T^{1/4}fw \in L^p(\mathbb{R})$, second, if $f$ be an absolutely continuous function with $f'w \in L^p(\mathbb{R})$. Then the de la Vallée Poussin mean $v_n(f)$ for a function $f$ gives one of the concrete example of (A). Moreover, if $f$ is more smoother function, $v_n(f)$ is not only a good approximation polynomial for $f$, but also its derivatives give an approximation for $f'$.
2. **Uniform convergence of the Fourier partial sum**: By the way, we show the condition uniformly convergence of $s_n(f)$ for a weight in a class $\mathcal{F}_\lambda(C^3+)$: Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$. Suppose that $f$ is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. If $f$ satisfies $\int_{\mathbb{R}} w(x)|df(x)| < \infty$, then

   $$\lim_{n\to\infty} \left\|(f - s_n(f))\frac{w}{T^{1/4}}\right\|_{L^\infty(\mathbb{R})} = 0.$$
3. **Lagrange interpolation polynomials**: Let $w_\rho$ be $w_\rho(x) := |x|^\rho w(x)$ for $\rho > 0$. For $f \in C(\mathbb{R})$, we write the Lagrange interpolation polynomial $L_n(f)(t)$ with nodes $\{t_{j,n,\rho}\}_{j=1}^n$, where $\{t_{j,n,\rho}\}_{j=1}^n$ are the zeros of $n$-th orthogonal polynomial with respect to $w_\rho$. We show the condition such that $L_n(f)(t)$ converges to $f$ with $w_\rho$ in $L^2$-norm: Let $f \in C(\mathbb{R})$ and $\beta > 1/2$. If $|(1 + |x|)^{\rho+\beta} w(x)f(x)| \to 0$ as $|x| \to \infty$, then we have

   $$\lim_{n\to\infty} \|(L_n(f) - f)w_\rho\|_{L^2(\mathbb{R})} = 0.$$