Plan of the study (Yosuke Saito)

As the author already said, it is expected that an difference deformation of the elliptic modulus arises in the elliptic Ruijsenaars system. However the difference deformation has not been realized yet. To study more on the elliptic Ruijsenaars system, the author has the following plan. For a complex number q which satisfies $|q|<1$, we define the q-infinite product $(x;q)_\infty$ $(x\in\mathbb{C})$ and the theta function $\Theta_q(x)$ $(x\in\mathbb{C}^\times)$ by

$$(x;q)_\infty:=\prod_{n\geq 0}(1-xq^n),\quad \Theta_q(x):=(q;q)_\infty(x;q)_\infty(qx^{-1};q)_\infty.$$

Then $\Theta_q(x)$ satisfies the heat equation $({D_x}^2-D_x)\Theta_q(x)=2D_q\Theta_q(x)$, where $D_z:=z\partial_z$. It can be understood that the heat equation tells us how q in $\Theta_q(x)$ is infinitesimally deformed. For complex numbers q, p satisfying $|q|<1$, $|p|<1$, we define the double infinite product $(x;q,p)_\infty$ $(x\in\mathbb{C})$ and the elliptic gamma function $\Gamma_{q,p}(x)$ $(x\in\mathbb{C}^\times)$ by

$$(x;q,p)_\infty:=\prod_{m,n\geq 0}(1-xq^mp^n),\quad \Gamma_{q,p}(x):=\frac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty}.$$

The elliptic gamma function satisfies the following difference equations.

$$\Gamma_{q,p}(qx)=\frac{\Theta_p(x)}{(p;p)_\infty}\Gamma_{q,p}(x),\quad \Gamma_{q,p}(px)=\frac{\Theta_q(x)}{(q;q)_\infty}\Gamma_{q,p}(x).$$

In the elliptic Ruijsenaars system, functions which are written by the elliptic gamma function usually appear. For the purpose of simplicity, for a complex number $t\in\mathbb{C}^\times$ we set $f(x):=\dfrac{\Gamma_{q,p}(tx)}{\Gamma_{q,p}(x)}$ $(x\in\mathbb{C}^\times)$. This satisfies $f(qx)=\dfrac{\Theta_p(tx)}{\Theta_p(x)}f(x)\cdots(*)$ and $f(x)\xrightarrow[t\to q]{}\dfrac{\Theta_p(x)}{(p;p)_\infty}$. Therefore, we can regard $(*)$ as a t-deformation of the heat equation for the theta function. Moreover, it is probable that a difference deformation of the elliptic modulus is found by considering the Taylor expansion in t around $t=q$ of the right hand side of $(*)$.