（英訳文）

We study the inverse Monge-Ampère flow (MA^{-1}-flow), a new parabolic flow on Fano manifolds and its self-similar solutions called Mabuchi solitons from the view point of Geometric Invariant Theory (GIT) in order to develop a new framework for dealing with K-unstable Fano manifolds.

Research A: We study geometric convergence properties along the flow. Concretely, we study the singular sets, dimension and algebraicity of the limit spaces.
Research B: We give a necessary sufficient condition for the existence of Mabuchi solitons in terms of GIT. Also we show that the existence of Mabuchi solitons is equivalent to that of Calabi's extremal metrics under some appropriate assumptions.

Research A

We study geometric convergence properties, and the singular sets, dimension, algebraicity of the limit spaces. The geometric convergence of the Kähler-Ricci flow (KRF) relies on Perelman's local non-collapsing theorem along the flow. However, we discovered that there is an obstruction "m_X" for the MA^{-1}-flow to have such a property. Since it is hard to consider the general situation from the beginning, we make assumptions that a Fano manifold (X, J) (J denotes a complex structure) satisfies a weak stability assumption, or have a nice symmetry as follows:

(A-1) The C^∞-colosure of the orbit $\mathrm{Diff}(X) \cdot J$ contains a KE structure J_∞ (adjacent condition). In particular, this condition yields the vanishing of m_X. We will show that the complex structure jumps at infinity along the MA^{-1}-flow, and converges to (X, J_∞) in C^∞-Cheeger Gromov sense.
(A-2) $X = \mathbb{P}(\mathcal{O} \oplus O_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Where $\mathcal{O}$ denotes the trivial bundle over $\mathbb{P}^2$. It is known that X is a K-unstable toric Fano 3-fold with $m_X \neq 0$, and a projective line bundles over $\mathbb{P}^2$. Then we will detect the subset $X_{\mathrm{sing}} \subset X$ where the flow will collapse and prove the local smooth convergence outside of X_{sing}.

From (A-1), (A-2), we can obtain examples of the flow in both cases $m_X = 0$, $m_X \neq 0$, which will be helpful for us to study the general situation.

Method of Research A

flow	parabolicity	necessary condition
$\{\omega_t\}$	○	K-stability
$\{J_t\}$	×	adjacent condition

Table 2 (dis)advantage of two flows

As for (A-1), it follows from Donaldson's GIT picture [Don15] and Moser's theorem, that the space of all Kähler metrics in $c_1(X)$ can be identified with the $\mathrm{Diff}(X) \cdot J$-orbit. With this correspondence, the MA^{-1}-flow $\{\omega_t\}$ is mapped into another flow $\{J_t\}$ on the moduli space of complex structures. An advantage of the flow $\{J_t\}$ is that it has a chance to converge under the adjacent condition (a weaker assumption than K-stability). A disadvantage is that it is invariant under the action of symplectic diffeomorphisms (in particular, it is never parabolic). On the other hand, $\{\omega_t\}$ had an (dis)advantage completely opposite to $\{J_t\}$. Since there flows are equivalent, we can prove the convergence of the flow $\{J_t\}$ by focusing on only the good points of them.

As for (A-2), we make good use the symmetry of X. Concretely, we focus on Kähler metrics that can be written as the sum of pullback from the Fubini-Study metric on the base $\mathbb{P}^2$ and rotationally invariant fiber metrics. Since this symmetry is preserved under the MA^{-1}-flow, we can reduced the flow to an evolution equation for functions f_t $(t \in [0,\infty))$ over an closed interval via the fiberwise moment map of the standard $U(1)$-action. Since X does not admit KE metrics, the fixed point of f_t, say f_∞ does not satisfy the positivity condition, or have a singularity along the end points. So we study the Lelong number of the singular metric corresponding to f_∞ to detect its singularity (conic, cusp, etc.) and estimate the volume that will be lost in the limit. With these observations, we detect the subset $X_{\mathrm{sing}} \subset X$ where the flow will collapse. On the other hand, since the equation of the MA^{-1}-flow is concave, we can apply the Evans-Krylov theory to get the higher order derivative estimates.

Research B

We give a necessary and sufficient condition for the existence of Mabuchi solitons. In [His19], it was proved that the existence of Mabuchi soliton is equivalent to the "uniform relative D-stability" (GIT stability). However since it is difficult to check this condition directly, we need a more practical criterion. Motivated by this, **we will show that we only have to check this condition for all special degenerations** (i.e. **a degeneration whose central fiber is $\mathbb{Q}$-Fano)**. In particular, since it is known that the existence of Calabi's extremal metrics implies this weakened version of relative D-stability, it also implies the existence of Mabuchi solitons.

Method of Research B

An effective tool to construct Kähler-Einstein metrics is Aubin-Yau's continuity path. We generalized this for Mabuchi solitons as follows:

$$\mathrm{Ric}(\omega_t) - (1-t)\hat{\omega} - t\omega_t = \sqrt{-1}\partial\bar{\partial}\log(1-\theta_t), \quad t \in [0,1], \tag{0.2}$$

where $\hat{\omega} \in c_1(X)$ is any reference metric, and θ_t is the Hamiltonian of the soliton vector fields with respect to ω_t. When $t=0$, the solution ω_0 of (0.2) solves the $(1-\theta_t)$-weighted Calabi-Yau equation, that has a unique smooth solution due to [BN14]. On the other hand, when $t=1$, ω_1 is nothing but a Mabuchi soliton. Thus, what we have to do is to solve (0.2) to $t=1$. For this, we may show that the Gromov-Hausdorff (GH) limit of (X,ω_t) is a $\mathbb{Q}$-Fano variety as $t \to T$, where T denotes the maximal existence time of the continuity path (0.2) (then we just apply the relative D-stability to obtain the solution for T, and hence T must be equal to 1). First we mention the non-collapsing property of the GH limit. From (0.2), we obtain the uniform lower bound for the Bakry-Emery Ricci tensor $\mathrm{Ric}(\omega_t) - \sqrt{1-}\partial\bar{\partial}\log(1-\theta_t)$. So we can apply the volume comparison and Myer's theorem (for Bakry-Emery Ricci curvature) to take a subsequence that converges to some compact length space (Z,d) (Gromov compactness theorem). As for the openness of the regular set of Z, we will apply the Anderson's result to the conformal metric $\tilde{g}_t := (1-\theta_t)^{1/n-1}g_t$ (where g_t denotes the Riemannian metric corresponding to ω_t). Indeed, the difference of $\mathrm{Ric}(\tilde{g}_t)$ and the Bakry-Emery Ricci tensor can be written as a function of θ_t, $\nabla\theta_t$, $\Delta_t\theta_t$. However, we can obtain the uniform control of these three quantities just by applying the maximum principle to (0.2). In particular, $\mathrm{Ric}(\tilde{g}_t)$ is uniformly bounded from below. In the KE setting, we can apply the ϵ-theorem for harmonic maps to obtain the uniform upper bound for the Ricci tensor along the continuity method [Szé16]. We consider whether the conformal analogue of [Szé16] still holds for the Ricci tensor $\mathrm{Ric}(\tilde{g}_t)$. Since (from the uniform bound for θ_t) $\tilde{g}_t$ and g_t are uniformly conformal equivalent, we can show that the GH limit of $(X, \tilde{g}_t)$ is isomorphic to that of Z. We also consider whether the ideas of Barkly-Emery/conformal geometry discussed above can be applied to get the argebraicity of Z, and estimates for the singular set and its dimension.