(英訳文)

Preface

I major in complex geometry and related topics. Especially, I am interested in a necessary and sufficiently condition for compact complex manifolds X with ample line bundles $L \to X$ to have canonical metrics. Usually a "canonical metric" means a metric with constant curvature, and this problem can be thought as a generalization of the classical uniformization theorem for compact Riemann surfaces. The equations for canonical metrics are given as PDEs, and satisfies some variational principles for energy functionals like harmonic forms. In particular, **geometric flows** obtained by the gradient flows of energy functionals play an important role to construct canonical metrics. While by using the **discretization method**, one can construct the approximation and study the convergence properties of them. Moreover, since X is projective, one can ask **the equivalence between the existence of canonical metrics and algebro-geometric stabilities**. Thus my research lies at the intersection of geometry, analysis and algebra, based on the rich geometric structures. In what follows, we will take a close look at each theme.

Research (1) Geometric flows

We say that X is Fano if the line bundle L is isomorphic to the anti-canonical line bundle $-K_X$. Then the **Kähler-Einstein (KE) condition** $\mathrm{Ric}(g) = g$ is one of the most natural curvature condition for Kähler metrics, and the Kähler-Ricci flow (KRF) is a very famous evolution equation designed to deform any Kähler metrics to the KE one. The KRF is studied by many experts and there are large amount of works so far. Motivated from this, I studied some extension of the KRF for general polarization L [Result 1], or non-compact manifolds [Result 11]. Moreover, I applied some techniques developed in the study of KRF to the mean curvature flow. As a results [Result 9], I showed that every holomorphic curve in a hyperkähler 4-manifold is linearly stable along the mean curvature flow. It seems that the linear stability for minimal submanifolds are known only in special cases so far (e.g. the ambient space is a KE surface [HL05], or flat torus [Li12]). In [Result 14], we introduced the new geometric flow named the tangent Lagrangian phase flow to construct the deformed Hermitian Yang-Mills (the mirror to a special Lagrangian) and studied the limiting behavior.

Research (2) Dynamical Systems

By using holomorphic sections on a large power L^k, one can embed X to a projective space (Kodaira embedding):

$$X \hookrightarrow \mathbb{P}(H^0(X, kL)^*) \simeq \mathbb{P}^{N_k}.$$

Then The space of all Fubini-Study metrics on $\mathbb{P}^{N_k}$ naturally isomorphic to a Riemannian symmetric space $\mathcal{H}_k \simeq GL(N_k; \mathbb{C})/U(N_k)$. It is known that by taking the limit $k \to \infty$, the space of all Kähler forms $\mathcal{H}$ can be approximated by $\mathcal{H}_k$:

$$\mathcal{H}_L = \overline{\bigcup_{k \geq 1} \mathcal{H}_k}.$$

Motivated by this, I constructed some discretization of solutions to PDEs (where the parameter is k) [Result 2, 4, 5, 12]. On the other hand, in [Result 8], we introduce a new dynamical system including the Ricci operator to construct canonical metrics.

Research (3) GIT stability

Roughly speaking, the space of Kähler metrics $\mathcal{H}$ has a good shape when X admits a canonical metric, and geometric flows have good convergence properties. On the other hand, from the view point of algebraic geometry, one study varieties (or, schemes in general). From the differential geometric side, this corresponds to studying the asymptotic behavior near the boundary $\partial\mathcal{H}$. Thus one can expect that there is a relationship between them. Indeed, I introduced some algebro-geometric notion of stabilities in [Result 2, 6], and studied the relation to canonical metrics and how to check the stability condition.

On the other hand, a concern in the unstable case is to construct **optimal destabilizers** which violate the stability condition. To attack this, one of the most effective way is to study the singularity formation along geometric flows. For instance, it is known that when $L = -K_X$, the KRF has a subsequence which converges to a "$\mathbb{Q}$-Fano variety" X_∞ (a Fano manifold with mild singularities) with a singular **Kähler-Ricci soliton** (KRS) (a self similar solution to the KRF) in the Gromov-Hausdorff (GH) sense [CW14].

In [Result 10], I studied the GH limit space X_∞ from the view point of non-Archimedean geometry and discovered some quantitative bound for the Donaldson-Futaki invariants on X_∞. In particular, when $X_\infty \simeq X$, combining with Kollár-Miyaoka-Mori [KMM92], I gave an affirmative answer to the finiteness conjecture of Futaki invariants on Kähler-Ricci solitons proposed by Phong-Song-Strum [PSS15].

The problem of finding optimal destabilizers dpends on what geometric flows we consider. However, the above results indicate that the singularity formation of the KRF is so mild that it never optimize the Donaldson-Futaki invariant. For this reason, We introduced in [Result 7] a new geometric flow named "**Inverse Monge-Ampère flow**" (MA^{-1}-flow):

$$\frac{\partial \omega(t)}{\partial t} = \sqrt{-1}\partial\bar{\partial}(1 - e^{\rho(\omega(t))}), \tag{0.2}$$

where $\rho(\omega)$ is a smooth function on X defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}\rho(\omega), \quad \int_X e^{\rho(\omega)}\omega^n = c_1(X)^n$$

Since the right hand side of (0.2) is computed as $e^{\rho(\omega(t))}(-\mathrm{Ric}(\omega(t)) + \omega(t) - |\nabla\rho(\omega(t))|^2)$, if $\rho(\omega(t))$ is close to zero in C^1, then the MA^{-1}-flow behaves like the KRF. Otherwise the behavior of both geometric flows are quite different. Indeed, we showd that the MA^{-1}-flow exists for all positive time. Moreover, restricted to the toric case, we also showed that along the flow, X collapses to a scheme whose number of irreducible components is less than 2, and the flow produces the optimal destabilizer in the sense that it achieves the infimum of the Ricci-Calabi functional:

$$R(\omega) := \int_X (1 - e^{\rho(\omega)})^2\omega^n, \quad \omega \in c_1(X)$$

at infinity. On the other hand, since every toric Fano manifold admits a KRS, the GH limit produced by the KRF is isomorphic to X and never attain the infimum of R. At this point, the study of MA^{-1}-flow has a different perspective from that of the KRF.

In [Result 13], we studied the limiting behavior of the line bundle mean curvature flow (the mirror to the Lagrangian mean curvature flow for graphs) introduced by Jacob-Yau [JY17] on Kähler surfaces X. As a consequence, we showed that there are finite number of holomorphic curves C_i ($i = 1, \dots, N$) of negative self-intersection such that the flow collapses on $\bigcup_{i=1}^N C_i$, and converges smoothly on $X \setminus \bigcup_{i=1}^N C_i$. This result indicates that the flow forms singularities in algebraic manner, which are not seen in the study of the original Lagrangian mean curvature flow. Then it is thought that the set $\bigcup_{i=1}^N C_i$ destabilize X algebraically in some sense.