On wild harmonic bundles

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Plan of the talk

(I) **Introduction**
- **Definition**
- **Corlette-Simpson correspondence**
- **Main issues in the study of tame and wild harmonic bundles**
- **Application to algebraic D-modules**

(II) **Overview of the study on wild harmonic bundles**

Definition of harmonic bundle (1)

X	:	complex manifold
$(E, \overline{\partial}_E, \theta)$	:	Higgs bundle on X
		i.e., $\theta \in \mathrm{End}(E) \otimes \Omega^1_X$, $\theta \circ \theta = 0$
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∂_E and $\theta^\dagger$ are determined by

$$\overline{\partial} h(u, v) = h\big(\overline{\partial}_E u, v\big) + h\big(u, \partial_E v\big) \qquad \Big(u, v \in C^\infty(X, E)\Big)$$

$$h(\theta u, v) = h(u, \theta^\dagger v)$$

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Definition

h is called pluri-harmonic, if the connection

$$\mathbb{D}^1 = \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$$

is flat. In that case, $(E, \overline{\partial}_E, \theta, h)$ is called harmonic bundle.

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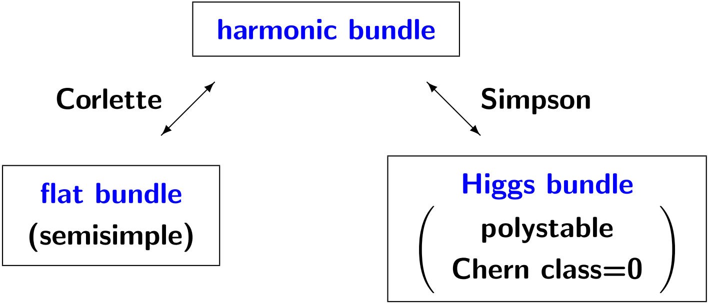
Corlette-Simpson correspondence

Corlette and Simpson established the following correspondence on any smooth projective variety:



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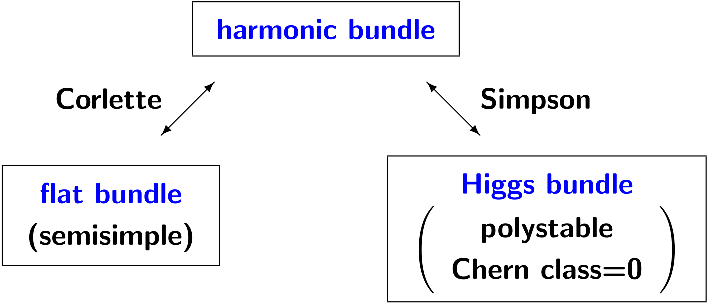

The tangent spaces of the moduli (the rank 1 case):

$$H^1(X, \mathbb{C}) \simeq H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1)$$

harmonic metric (Corlette)

$$\begin{array}{rcl} X & : & \text{Riemannian manifold,} \\ (V, \nabla) & : & \text{flat bundle} \\ h & : & \text{metric of } V \end{array}$$

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\Phi_h} & \{\text{hermitian metric}\} \\ \big\downarrow & & \\ X & & \end{array}$$

$$h \text{ harmonic} \overset{\text{def}}{\Longleftrightarrow} \Phi_h \text{ harmonic}$$

$$X\text{: compact Kahler} \Longrightarrow h \text{ pluri-harmonic}$$

Variation of polarized Hodge structure

X	:	complex manifold
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F	:	filtration by holomorphic subbundles $F^i \subset F^{i-1}$
S	:	flat pairing of V

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A typical example of a Hodge bundle

$$\mathcal{O}_X \oplus \Theta_X, \quad \theta_X : \mathcal{O}_X \longrightarrow \Theta_X \otimes \Omega_X^1$$

Deformation to VPHS

$$\big(\boldsymbol{E},\boldsymbol{\theta}\big) \rightsquigarrow \big(\boldsymbol{E},\boldsymbol{\alpha}\,\boldsymbol{\theta}\big)\ (\boldsymbol{\alpha}\in\mathbb{C}^{\times})$$ **obvious deformation**

$$\Downarrow$$

$$\big(\boldsymbol{V},\boldsymbol{\nabla}\big) \rightsquigarrow \big(\boldsymbol{V_\alpha},\boldsymbol{\nabla_\alpha}\big)\ (\boldsymbol{\alpha}\in\mathbb{C}^{\times})$$ **non-trivial deformation**

$\exists \lim_{\boldsymbol{\alpha}\to 0}(\boldsymbol{V_\alpha},\boldsymbol{\nabla_\alpha})$ **underlies a variation of polarized Hodge structures**

Deformation to VPHS

Proposition (Simpson)

$\mathbf{SL}(n, \mathbb{Z})$ $(n \geq 3)$ *cannot be the fundamental group of a smooth projective variety.*

- (V, ∇) **underlies a VPHS** $\Longrightarrow$ **The real Zariski closure of** $\pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ **is "of Hodge type".**
- $\mathrm{SL}(n, \mathbb{Z})$ **is rigid.**
- $\mathrm{SL}(n, \mathbb{R})$ **is not of Hodge type.**

Flat bundle with a non-trivial deformation

X : projective manifold

(V, ∇) : flat bundle on X.

Theorem (Simpson)

Assume $\operatorname{rank} V = 2$. *If* (V, ∇) *has a non-trivial deformation,*

- $\exists (V', \nabla')$*: a flat bundle on a projective curve* C.
- $\exists F : X \longrightarrow C$
- $(V, \nabla) = F^*(V', \nabla')$.

Theorem (Reznikov)

$c_i(V) = 0$ $(i > 1)$ *in the Deligne cohomology group of* X.

Tame and wild harmonic bundles

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harmonic bundle		flat bundle
wild	$\Longleftrightarrow$	meromorphic
tame		regular singular

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Definition

- $(E, \overline{\partial}_E, \theta, h)$ *is tame, if* $a_j(z)$ *are holomorphic on* Δ.
- $(E, \overline{\partial}_E, \theta, h)$ *is wild, if* $a_j(z)$ *are meromorphic on* Δ.

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Remark

In the higher dimensional case, we need more complicated condition for wildness.

Tame harmonic bundles

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- (A1) **Prolongation**
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- (B) **Kobayashi-Hitchin correspondence**
  (Generalization of Corlette-Simpson correspondence)
- (C) **Polarized (regular) pure twistor D-module**
  - (C1) **Hard Lefschetz theorem**
  - (C2) **Correspondence between tame harmonic bundles and polarized pure twistor D-modules**

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(D) **Application to algebraic D-modules**

(Sabbah's program)

Wild harmonic bundle

(A) **Asymptotic behaviour of wild harmonic bundles**

- (A1) **Prolongation**
- (A2) **Reduction**

(B) **Algebraic meromorphic flat bundles and Higgs bundles**

- (B1) **Kobayashi-Hitchin correspondence**
- (B2) **Characterization of semisimplicity**
  Resolution of turning points

(C) **Polarized wild pure twistor D-modules**

- (C1) **Hard Lefschetz Theorem**
- (C2) **Correspondence between polarized wild pure twistor D-modules and wild harmonic bundles**

(D) **Application to algebraic D-modules**

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f : projective morphism $X \longrightarrow Y$

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We obtain the push-forward

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and the holonomic $\mathcal{D}_Y$-modules

$$f_\dagger^m \mathcal{F} := \text{the } m\text{-th cohomology of } f_\dagger \mathcal{F}$$

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II. Overview of the study on wild harmonic bundles

- **(B2) Characterization of semisimplicity**
  Resolution of turning points
- **(C) Polarized wild pure twistor D-modules**
- **(B)+(C) $\Longrightarrow$ Application to algebraic D-modules**
- **(A) Asymptotic behaviour of wild harmonic bundles**

II. Overview of the study on wild harmonic bundles

(B) **Algebraic meromorphic flat bundles**

Higgs bundles

λ-flat bundles

(B1) **Kobayashi-Hitchin correspondence**

(B2) **Characterization of semisimplicity**

Resolution of turning points

Characterization of semisimplicity

Let X be a complex smooth projective variety.

Proposition (Corlette)

For any flat bundle on X, the following two conditions are equivalent.

- *It is semisimple, i.e., a direct sum of irreducible ones.*
- *It has a pluri-harmonic metric.*

Such a pluri-harmonic metric is essentially unique.

Characterization of semisimplicity

Let D be a normal crossing divisor of X.

Proposition

Such a characterization was generalized for any meromorphic flat bundle on (X, D) with regular singularity. (The pluri-harmonic metric h of $(\mathcal{E}, \nabla)_{|X-D}$ should satisfy some condition around D.)

$\dim X = 1$ **essentially due to Simpson with Sabbah's observation that semisimplicity is related to parabolic polystability.**

$\dim X \geq 2$ **two known methods**

- **Jost-Zuo (with a minor complement by M)**
- **Use Kobayashi-Hitchin correspondence (M)**

Characterization of semisimplicity

Theorem (B2.1)

We can establish such a characterization even in the non-regular case.

wild harmonic bundle $\longleftrightarrow$ *semisimple meromorphic flat bundle*

$\dim X = 1$ **Sabbah (a related work due to Biquard-Boalch)**

$\dim X \geq 2$ **M.**

We have a serious difficulty caused by the existence of turning points in the higher dimensional case.

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$$\varphi^*(\mathcal{E}, \nabla)_{|\widehat{O}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{\mathcal{E}}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$$

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- $\mathrm{Irr}(\nabla) \subset \mathcal{O}_\Delta(*O)$, finite subset. (It is well defined in $\mathbb{C}((z))/\mathbb{C}[\![z]\!] \simeq z^{-1}\mathbb{C}[z^{-1}]$.)
- $\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}$ has regular singularity for each $\mathfrak{a}$.

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closed analytic subset $\quad Z \subset \Delta^{n-1}$

ramified covering $\quad \varphi : (\Delta, O) \times \Delta^{n-1} \longrightarrow (\Delta, O) \times \Delta^{n-1}$

such that $\varphi^*(\mathcal{E}, \nabla)_{|\widehat{O} \times (\Delta^{n-1} \setminus Z)}$ locally has such a nice decomposition. (More strongly, Malgrange showed the existence of Deligne-Malgrange lattice.)

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Definition

The points of Z are called turning points. (It can be defined appropriately even in the case of normal crossing poles.)

Example of turning points

Take a meromorphic flat bundle $(\mathcal{E}, \nabla)$ on $\mathbb{P}^1$ such that (i) 0 is the only pole of $(\mathcal{E}, \nabla)$, (ii) it has non-trivial Stokes structure. For example,

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(*0)\, v_1 \oplus \mathcal{O}_{\mathbb{P}^1}(*0)\, v_2$$

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Let $F : \mathbb{C}^2 \longrightarrow \mathbb{P}^1$ be a rational map given by $F(x, y) = [x : y]$. The pole of $F^*(\mathcal{E}, \nabla)$ is $\{x = 0\}$, and it can be shown that $(0, 0)$ is a turning point.

Difficulty caused by the existence of turning points

The existence of turning points prevents us from applying Kobayashi-Hitchin correspondence to a characterization of semisimplicity.

Difficulty caused by the existence of turning points

A general framework in global analysis:

(i) **Take an appropriate metric of $(\mathcal{E}, \nabla)_{|X-D}$. (Some finiteness condition on the curvature.)**

(ii) **Deform it along the heat flow.**

(iii) **The limit of the flow should be a Hermitian-Einstein metric, and under some condition, it should be a pluri-harmonic metric.**

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Remark

Even if there are no turning points, we need some trick.

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Step 1 **Take the Hukuhara–Levelt–Turrittin decomposition after ramified covering.**

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We have two steps to understand the structure of a meromorphic flat bundle on a curve.

- Step 1 **Take the Hukuhara–Levelt–Turrittin decomposition after ramified covering.**
- Step 2 **Lift it to flat decompositions on small sectors.** **($\Longrightarrow$ Stokes structure)**

Difficulty caused by the existence of turning points

The existence of turning points is a serious difficulty for a general theory of asymptotic analysis of meromorphic flat bundles studied by Majima and Sabbah.

We have two steps to understand the structure of a meromorphic flat bundle on a curve.

- Step 1 **Take the Hukuhara–Levelt–Turrittin decomposition after ramified covering.**
- Step 2 **Lift it to flat decompositions on small sectors. ($\Longrightarrow$ Stokes structure)**

Briefly speaking, they established the higher dimensional version of Step 2.

Sabbah's conjecture

We hope to have a resolution of turning points.

Sabbah's conjecture

We hope to have a resolution of turning points.

Sabbah established it in the case $\dim X = 2$, $\mathrm{rank}(\mathcal{E}, \nabla) \leq 5$.

Resolution of turning points

Theorem (B2.2)

Let X be a smooth proper algebraic variety, and let D be a normal crossing hypersurface. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D).

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Let X be a smooth proper algebraic variety, and let D be a normal crossing hypersurface. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Then, there exists a projective birational morphism $\varphi : (X', D') \longrightarrow (X, D)$ such that $\varphi^*(\mathcal{E}, \nabla)$ has no turning points.

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It seems of foundational importance in the study of algebraic meromorphic flat bundles or algebraic holonomic D-modules, and it might be compared with the existence of a resolution of singularities for algebraic varieties.

Resolution of turning points

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Let X be a smooth proper algebraic variety, and let D be a normal crossing hypersurface. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Then, there exists a projective birational morphism $\varphi : (X', D') \longrightarrow (X, D)$ such that $\varphi^*(\mathcal{E}, \nabla)$ has no turning points.

It seems of foundational importance in the study of algebraic meromorphic flat bundles or algebraic holonomic D-modules, and it might be compared with the existence of a resolution of singularities for algebraic varieties.

Remark

Kedlaya established the existence of resolution of turning points for any meromorphic flat bundle on any general complex surface!

Brief sketch of the proof

Theorem (B2.1)

Characterization of semisimplicity of algebraic meromorphic flat bundles by the existence of nice pluri-harmonic metrics.

Theorem (B2.2)

Existence of resolution of turning points for algebraic meromorphic flat bundles.

Brief sketch of the proof

Thm B2.2 $\dim X = 2$ **mod p-reduction and p-curvatures**

(We may also apply Kedlaya's result.)

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Thm B2.1 $\dim X \geq 3$ — **Mehta-Ramanathan type theorem**

Brief sketch of the proof

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Thm B2.2 $\dim X \geq 3$ — **Reduced to the case $(\mathcal{E}, \nabla)$ is simple**

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Thm B2.2 $\dim X \geq 3$	**Reduced to the case $(\mathcal{E}, \nabla)$ is simple** $\Longrightarrow$ **the associated Higgs field θ**

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turning points for $(\mathcal{E}, \nabla)$
= "turning points for θ"

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turning points for $(\mathcal{E}, \nabla)$
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We can use classical techniques in complex geometry.

Brief sketch of the proof

We use the theory of polarized wild pure twistor D-modules for non-projective case.

- **Take a birational morphism $\varphi : X' \longrightarrow X$ such that X' is projective.**
- **Take a nice pluri-harmonic metric for $\varphi^*(\mathcal{E}, \nabla)$.**
- **Use the Hard Lefschetz theorem to obtain a nice pluri-harmonic metric for $(\mathcal{E}, \nabla)$.**

(C) **Polarized wild pure twistor D-modules**

(C1) **Hard Lefschetz Theorem**

(C2) **Correspondence between polarized wild pure twistor D-modules and wild harmonic bundles**

What is a polarized wild pure twistor D-module?

Briefly and imprecisely,

Polarized wild pure twistor D-module	$\doteqdot$	D-module with pluri-harmonic metric

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Polarized wild pure twistor D-module	$\doteqdot$	D-module with pluri-harmonic metric

How to define "pluri-harmonic metric" for D-modules?

A very important hint was given by Simpson!

Mixed twistor structure

harmonic bundle $\overset{\text{similarity}}{\longleftrightarrow}$ **variation of polarized Hodge structure**

Mixed twistor structure

harmonic bundle	$\overset{\text{similarity}}{\longleftrightarrow}$	variation of polarized Hodge structure

- **A variation of polarized Hodge structure has the underlying harmonic bundle.**
- **The isomorphism between the de Rham cohomology and the Dolbeault cohomology (the cohomology group associated to the Higgs bundle).**

Mixed twistor structure

To establish this similarity in the level of definitions, Simpson introduced the notion of mixed twistor structure.

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Naive Hope:

Statement, Proof for Hodge structure

Mixed twistor structure

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Naive Hope:

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$\Downarrow$ **Replace "Hodge" with "Twistor"**

Statement, Proof for Twistor structure

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- **"polarization" can be defined appropriately.**

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- "**polarization**" can be defined appropriately.
- **harmonic bundle $=$ variation of polarized pure twistor structure**

We can formulate "harmonic bundle version" or "twistor version" of most objects in the theory of variation of Hodge structure.

Polarized wild pure twistor D-modules

Polarized wild pure twistor D-module	$\doteqdot$	holonomic D-module with pluri-harmonic

Morihiko Saito

polarized pure Hodge module $\doteqdot$ D-module + PHS

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Sabbah **introduced wild polarized pure twistor D-module as a twistor version. It was still a hard work, and he made various innovations and observations such as sesqui-linear pairings, their specialization by using Mellin transforms, the nearby cycle functor with ramification and exponential twist for $\mathcal{R}$-triples, and so on.**

Hard Lefschetz Theorem

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The following theorem is essentially due to Saito and Sabbah.

Theorem (Hard Lefschetz Theorem)

Polarizable wild pure twistor $\boldsymbol{D}$-modules have nice functorial property for push-forward via projective morphisms.

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Let $f : X \longrightarrow Y$ be a projective morphism.

polarizable wild pure twistor $\mathcal{D}_X$-modules	$\Longrightarrow$	**polarizable wild pure twistor $\mathcal{D}_Y$-modules**
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Moreover, for a line bundle L on X, ample relative to f, the following induced morphisms are isomorphisms

$$c_1(L)^j : f_{\dagger}^{-j}\mathcal{T} \xrightarrow{\simeq} f_{\dagger}^{j}\mathcal{T} \otimes \mathbb{T}^S(j)$$

Wild harmonic bundles and polarized wild PTD

Theorem

On a complex manifold X, we have the following correspondence

$$\boxed{\textit{Wild harmonic bundle}} \iff \boxed{\textit{Polarized wild pure twistor } D\textit{-module}}$$

Wild harmonic bundles and polarized wild PTD

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On a complex manifold X, we have the following correspondence

$$\boxed{\textit{Wild harmonic bundle}} \Longleftrightarrow \boxed{\textit{Polarized wild pure twistor } D\textit{-module}}$$

- **Let Z be an irreducible closed analytic subset of X, and let U be a smooth open subset of Z which is the complement of a closed analytic subset of Z. Then, any wild harmonic bundle on U is extended to polarized wild pure twistor D-module on Z.**

Wild harmonic bundles and polarized wild PTD

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- **Any polarized wild pure twistor D-module is the direct sum of minimal extensions.**

II. Overview of the study on wild harmonic bundles

(B)+(C) $\Longrightarrow$ Application to algebraic D-modules

Application to algebraic D-modules

Theorem

On a smooth projective variety X, we have the following correspondence through wild harmonic bundles

semisimple holonomic D-modules	$\Longleftrightarrow$	*polarizable wild pure twistor D-module*

Application to algebraic D-modules

Theorem

On a smooth projective variety X, we have the following correspondence through wild harmonic bundles

$$\boxed{\begin{array}{c}\textit{semisimple}\\ \textit{holonomic } D\textit{-modules}\end{array}} \iff \boxed{\begin{array}{c}\textit{polarizable wild}\\ \textit{pure twistor } D\textit{-module}\end{array}}$$

$\Longrightarrow$ **We obtain HLT for algebraic semisimple holonomic D-modules from HLT for polarizable wild pure twistor D-modules.**

II. Overview of the study on wild harmonic bundles

(A) **Asymptotic behaviour of wild harmonic bundles**

(A1) **Prolongation**

(A2) **Reduction**

Underlying λ-flat bundles

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Underlying λ-flat bundles

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Prolongation

Let X be a complex manifold, and let D be a normal crossing hypersurface of X. From $(E, \overline{\partial}_E, \theta, h)$ on $X - D$, we obtain λ-flat bundle $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ on $X - D$:

$$\mathcal{E}^\lambda = \big(E, \overline{\partial}_E + \lambda\theta^\dagger\big), \quad \mathbb{D}^\lambda = \overline{\partial}_E + \lambda\theta^\dagger + \lambda\partial_E + \theta$$

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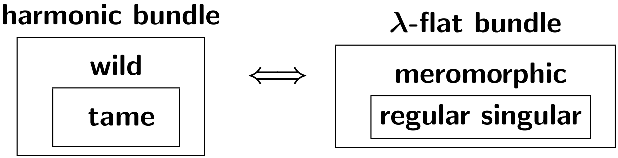
First goal **We would like to prolong it to a meromorphic λ-flat bundle on (X, D) with good lattices.**



Prolongation

Let $X := \Delta^n$, $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X - D$. We have the associated λ-flat bundle $(\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$ on $X - D$.

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$$\mathcal{P}\mathcal{E}^{\lambda}(U) := \Big\{ f \in \mathcal{E}^{\lambda}(U \setminus D) \,\Big|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-N}\Big)\ \exists N > 0 \Big\}$$

$$\mathcal{P}_0\mathcal{E}^{\lambda}(U) := \Big\{ f \in \mathcal{E}^{\lambda}(U \setminus D) \,\Big|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-\epsilon}\Big)\ \forall \epsilon > 0 \Big\}$$

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By taking the sheafification, we obtain the $\mathcal{O}_X(*D)$-module $\mathcal{P}\mathcal{E}^{\lambda}$ and the $\mathcal{O}_X$-module $\mathcal{P}_0\mathcal{E}^{\lambda}$.

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Theorem

- $(\mathcal{P}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ *is a good meromorphic* λ*-flat bundle.*
- $\mathcal{P}_0\mathcal{E}^\lambda$ *is locally free, and "good lattice".*

Outline of a part of the proof

Some steps to show that $\mathcal{P}_0\mathcal{E}^{\lambda}$ is locally free.

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- **The estimate for the Higgs field θ (the wild version of Simpson's main estimate).**
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- The estimate for the Higgs field θ (the wild version of Simpson's main estimate).
  - Asymptotic orthogonality of "generalized eigen decomposition"
  - Boundedness of the "nilpotent parts"
- We can show that $(\mathcal{E}^{\boldsymbol{\lambda}}, h)$ is acceptable, i.e., the curvature of $(\mathcal{E}^{\boldsymbol{\lambda}}, h)$ is bounded with respect to h and the Poincaré metric of $X - D$.
- We have developed a general theory of acceptable bundles, i.e., any acceptable bundles are naturally extended to locally free sheaves by the above procedure. Hence, $\mathcal{P}_0\mathcal{E}^{\boldsymbol{\lambda}}$ is locally free.

Prolongation

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We need and have something more.

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Second Goal **We should consider the prolongation of the family of λ-flat bundles.**

Prologation

Theorem

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- *$\mathcal{P}_0\mathcal{E}^{\boldsymbol{\lambda}}$ is locally free, and "good lattice".*

We need and have something more.

Second Goal **We should consider the prolongation of the family of λ-flat bundles. Because $\{\mathcal{P}\mathcal{E}^{\boldsymbol{\lambda}} \,|\, \lambda \in \mathbb{C}\}$ cannot be a nice meromorphic object, we have to think the deformation of meromorphic λ-flat bundles caused by the variation of irregular values.**

Prolongation: Stokes filtration in the curve case

Let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection on (Δ, O), which is unramified. The formal decomposition

$$(\mathcal{E}, \nabla)_{|\widehat{O}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{\mathcal{E}}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$$

can be lifted to a flat decomposition on each small sector S of Δ^*:

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The filtration (Stokes filtration, or Deligne-Malgrange filtration)

$$\mathcal{F}^S_{\mathfrak{a}} = \bigoplus_{\mathfrak{b} \leq_S \mathfrak{a}} \mathcal{E}_{\mathfrak{b},S} \quad \mathfrak{b} \leq_S \mathfrak{a} \Longleftrightarrow -\operatorname{Re}(\mathfrak{b}) \leq -\operatorname{Re}(\mathfrak{a}) \text{ on } S$$

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is canonically determined (some compatibility condition). We can recover $(\mathcal{E}, \nabla)$ from $(\mathcal{E}, \nabla)_{|X-D}$ and $\{\mathcal{F}^S \mid S \subset \Delta^*\}$ (Deligne, Malgrange inspired by the work of Sibuya).

Prolongation: Deformation

For any $T > 0$, we set $\mathrm{Irr}(\nabla^{(T)}) := \left\{ T\mathfrak{a} \,\middle|\, \mathfrak{a} \in \mathrm{Irr}(\nabla) \right\}$, and

$$\mathcal{F}^{(T)\,S}_{T\mathfrak{a}} := \mathcal{F}^{S}_{\mathfrak{a}}$$

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Applying similar procedure to $(\mathcal{PE}^{\lambda}, \mathbb{D}^{\lambda})$ with $T = (1 + |\lambda|^2)^{-1}$, we obtain $(\mathcal{QE}^{\lambda}, \mathbb{D}^{\lambda})$.

Theorem

The family $\{(\mathcal{QE}^{\lambda}, \mathbb{D}^{\lambda}) \mid \lambda \in \mathbb{C}\}$ *gives a nice meromorphic object.*

Prolongation

- We need and have something more (the parabolic structure, the eigenvalues of the residues, the irregular decomposition).
- Kobayashi-Hitchin correspondence.
- Characterization of semisimplicity.
- Resolution of turning points

Reductions

We would like to understand more detailed property.

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wild

$\Downarrow$ **Gr w.r.t. Stokes structure**

tame

$\Downarrow$ **Gr w.r.t. KMS structure**

polarized mixed twistor structure

$\Downarrow$ **Gr w.r.t. weight filtration**

polarized mixed twistor structure of split type

Reductions of meromorphic flat bundle on curve

It can be compared with the following very simple reductions for meromorphic flat bundles on a curve satisfying unramifiedness condition.

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meromorphic (irregular)

$\Downarrow$

meromorphic (regular)

$\Downarrow$

vector space + nilpotent endomorphism

$\Downarrow$

vector space + nilpotent endomorphism (graded)

Reductions of meromorphic flat bundle on curve

- **The first reduction** is taking a direct summand in the Hukuhara–Levelt–Turrittin decomposition

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or we prefer to regard it as Gr with respect to Stokes structure.

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- **The third reduction** is Gr with respect to the weight filtration.

Reductions

- Relations among the weight filtrations.
- Norm estimate, i.e., a wild pluri-harmonic metric is determined by the residues and the parabolic structures, up to boundedness.
- Correspondence between wild harmonic bundles and polarized wild pure twistor D-modules.
- Vanishing of characteristic numbers (Kobayashi-Hitchin correspondence).