Some criteria for quasiconformality and Landau-Bloch constants of harmonic mappings in the unit disk

Abstract. Univalent harmonic mappings in the plane are natural generalizations of conformal mappings. In 1984, J.Clunie and T.Sheil-Small published a landmark paper which pointed out that many of the classical results for conformal mappings have clear analogues for univalent harmonic mappings. Since that time the subject has developed rapidly and a number of basic interesting results have been obtained. Especially it has been discussed by many mathematicians to find some necessary and sufficient conditions for harmonic mappings to be quasiconformal under certain conditions. A quasiconformal mapping, introduced by H.Grötzsch (1928) and named by L.Ahlfors (1935), is a generalized homeomorphic solution of a Beltrami equation. It has many geometric and analytic properties.

In this talk, we give some criteria for quasiconformality and Landau-Bloch constants of harmonic mappings in the unit disk.

Let $w(z)$ be a univalent harmonic mapping of the unit disk U onto itself with $w(0) = 0$. In 1952, E.Heinz proved the so-called Heinz's inequality which gives an absolute positive lower bound of $|w_{\bar{z}}(0)|^2+|w_z(0)|^2$. Under the additional assumption that $w(z)$ is a K-quasiconformal mapping, we obtain sharp estimates of (a variant of) Heinz's inequality which improve D.Partyka and K.Sakan's results.

Assume that $\Omega \subseteq \mathbb{C}$ is a bounded convex domain, and that $F : T \mapsto \Gamma = \partial\Omega$ is a sense-preserving homeomorphism of the unit circle T onto the boundary of Ω. According to Radó-Kneser-Choquet theorem we know that the Poisson integral $w = P[F](z)$ of F is a univalent harmonic mapping of U onto Ω. We give a criterion for quasiconformality of $w = P[F](z)$ by means of some conditions on F. This result partially improves D.Kalaj's theorem. If $w = P[F](z)$ is a univalent harmonic mapping of the unit disk U onto itself, then we obtain an estimate for the Jacobian of w. As an application, the maximal dilatation of w and $|\partial w|$ are also estimated, when it is a harmonic quasiconformal self-mapping of the unit disk U.

For a bounded harmonic mapping defined in the unit disk U, it is important to estimate its Landau-Bloch constants. For a bounded domain $G \subseteq \mathbb{C}$ including 0, suppose that f is a harmonic mapping of U into G which satisfies $f(0) = f_{\bar{z}}(0) = f_z(0) - 1 = 0$ and the coefficients condition

$$|a_n| + |b_n| \leq c, \quad n = 2, 3, \cdots$$

for some positive constant c. Then we estimate Landau-Bloch constants for both f and $L(f)$, where $L(f)(z) := zf_z(z) - \bar{z}f_{\bar{z}}(z) = -if_\theta(z)$. Moreover, if f is a harmonic K-quasiconformal mapping, we also obtain a univalent radius of $L(f)$ by means of K.

Finally we introduce two subclasses of harmonic mappings and discuss quasiconformality of these two classes. This also gives some criteria for quasiconformality of harmonic mappings.