A HEEGAARD FLOER HOMOLOGY FOR BIPARTITE SPATIAL GRAPHS AND ITS PROPERTIES

YUANYUAN BAO

Abstract. We discuss a way to define a Heegaard Floer homology for bipartite spatial graphs, and give some properties for this homology.

1. Introduction

The Heegaard Floer homology for knots and links defined by Ozsváth and Szabó [5, 7], and independently Rasmussen [8] has made great impacts on the study of knot theory. In this paper, we attempt to extend their idea to define a Heegaard Floer homology for spatial graphs, which are generalizations of knots and links. Manolescu, Ozsváth and Thurston [3] redefined the Heegaard Floer homology for links in the 3-sphere S^3. They used grid diagrams for links instead of Heegaard diagrams, and their definition is combinatorial. Extending the idea in [3], Harvey and O'Donnol announced that they have defined a combinatorial Floer homology for spatial graphs in S^3. The preprint of their work is not available yet.

Our definition in this paper is only for balanced bipartite spatial graphs in a closed oriented 3-manifold. It turns out that, just as the Heegaard Floer homology for links, our definition is a special case of sutured Floer homology defined by Juhász [2]. Therefore it will be helpful to compare with [2] when reading this paper. We also use many facts about multi-pointed Heegaard diagrams established in [7].

Heegaard Floer type homology is basically defined on a Heegaard diagram. In Section 2, we define the Heegaard diagram for a bipartite spatial graph. Then in the case the ground 3-manifold is S^3, we provide a method for constructing a Heegaard diagram for a balanced bipartite spatial graph G from a proper graph projection of G on S^2. In Section 3, we define our Heegaard Floer homology for balanced bipartite spatial graphs. In Section 4, some properties of this homology are given.

2. Heegaard diagrams

Definition 2.1. A graph G is called a *bipartite graph* if its vertex set V is a disjoint union of two non-empty sets V_1 and V_2 so that there is no edge connecting vertices from the same V_i for $i = 1, 2$. We denote the graph by G_{V_1,V_2}. If $|V_1| = |V_2|$, G is called *balanced*.

In this paper, we do not consider those graphs with isolated vertices.

The splitting $V = V_1 \coprod V_2$ is not necessarily unique. But when G is connected, it is very easy to see the following lemma.

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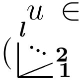
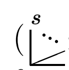
Lemma 2.2. *If a bipartite graph G is connected, then the choice of V_1 and V_2 is unique without considering the order of V_1 and V_2.*

Given a compact closed oriented 3-manifold M, we can consider a smooth embedding of a graph $G = (V, E)$ into M, the vertices of V corresponding to points in M and the edges of E corresponding to pairwise disjoint simple arcs in M. We call the isotopy class of such an embedding a spatial graph in M and still denote it by G if no confusion is caused. All manifolds considered below are assumed to be compact and oriented and we work in the P.L. category.

Definition 2.3. Suppose G_{V_1,V_2} is a balanced bipartite spatial graph in a closed 3-manifold M. A quartet $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ is called a Heegaard diagram for G_{V_1,V_2} if it satisfies the following.

(i) Σ is an oriented genus g closed surface, which is called the Heegaard surface. $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \cdots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots, \beta_d\}$ are d_1-tuple and d_2-tuple of oriented simple closed curves on Σ respectively, and $\boldsymbol{z} = \{z_1, z_2, \cdots, z_m\}$ is an m-tuple of points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, called basepoints. Here $|V_i| = d_i - g + 1$ for $i = 1, 2$ and m is the number of edges of G_{V_1,V_2}.

(ii) Attaching 2-handles to $\Sigma \times [-1, 0]$ (resp. $\Sigma \times [0, 1]$) along $\boldsymbol{\alpha} \subset \Sigma \times \{-1\}$ (resp. $\boldsymbol{\beta} \subset \Sigma \times \{1\}$), we get a $(d_1 - g + 1)$-punctured (resp. $(d_2 - g + 1)$-punctured) handlebody which we call $U_{\boldsymbol{\alpha}}$ (resp. $U_{\boldsymbol{\beta}}$). We cap off the 2-sphere components of $U_{\boldsymbol{\alpha}}$ and $U_{\boldsymbol{\beta}}$, except Σ in the case $\Sigma = S^2$, to get handlebodies $\overline{U_{\boldsymbol{\alpha}}}$ and $\overline{U_{\boldsymbol{\beta}}}$ respectively. Then $\overline{U_{\boldsymbol{\alpha}}} \bigcup_{\Sigma \times \{0\}} \overline{U_{\boldsymbol{\beta}}}$ is the 3-manifold M. The orientation of Σ is induced from that of $U_{\boldsymbol{\alpha}}$, which in turn is inherited from that of M.

(iii) For each vertex $v \in V_1$ (resp. $u \in V_2$) whose valency is l (resp. s), there is a smooth embedding φ_v : (, $\{1, 2, \cdots, l\}) \hookrightarrow (\Sigma \backslash \boldsymbol{\alpha}, \boldsymbol{z})$ (resp. ψ_u : (, $\{1, 2, \cdots, s\}) \hookrightarrow (\Sigma \backslash \boldsymbol{\beta}, \boldsymbol{z})$). The images of φ_{v_i} for $v_i \in V_1$ (resp. ψ_{u_j} for $u_j \in V_2$) are disjoint from each other except at endpoints. Moreover, $(\bigcup_{v_i \in V_1} \mathrm{Im}(\varphi_{v_i})) \cup (\bigcup_{u_j \in V_2} \mathrm{Im}(\psi_{u_j}))$ is G_{V_1,V_2}. Here we push $\mathrm{Im}(\varphi_{v_i})$ (resp. $\mathrm{Im}(\psi_{u_j})$) slightly into $\overline{U_{\boldsymbol{\alpha}}}$ (resp. $\overline{U_{\boldsymbol{\beta}}}$).

Any balanced bipartite graph has such Heegaard diagrams. When G_{V_1,V_2} is balanced, namely $|V_1| = |V_2|$, we naturally have $d_1 = d_2$.

In the case $M \cong S^3$ we extend the idea in [4] and provide an algorithm to construct a Heegaard diagram for a balanced bipartite graph from its graph projection on the 2-sphere S^2. Consider a spatial diagram $D \subset S^2$ for a given balanced bipartite spatial graph $G_{V_1,V_2} \subset S^3$. We assume that D is connected.

(i) Take a tubuler neighbourhood of D in S^3. It is a handlebody and its boundary is the Heegaard surface Σ.

(ii) For each crossing of D, introduce an $\boldsymbol{\alpha}$-curve following the rule in Figure 1 (B). The diagram D separates S^2 into several regions. Choose a region and call it β_0. For each region except β_0, introduce a $\boldsymbol{\beta}$-curve which spans the region as in Figure 1 (A).

(iii) For each vertex $u \in V_2$ with valency l, introduce $l-1$ $\boldsymbol{\alpha}$-curves and l basepoints. Introduce a $\boldsymbol{\beta}$-curve for all but one vertices in V_2, which enclose all the base points at that vertex. See Figure 1 (C).

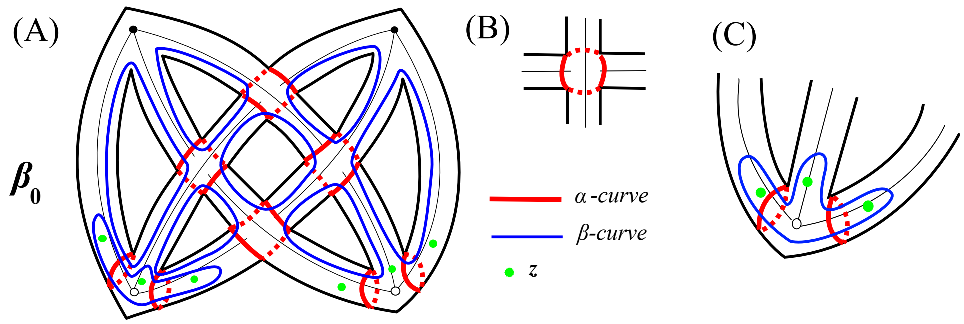


FIGURE 1. The Heegaard diagram associated with a graph projection.

By a standard argument in Morse theory, we have the following theorem. The proof is analogous to that of [2, Proposition 2.15].

Theorem 2.4. *Two Heegaard diagrams* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ *and* $(\Sigma', \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{z'})$ *for a given balanced bipartite spatial graph* G_{V_1,V_2} *can be connected by a finite sequence of the following moves:*

(i) *Isotopies of* $\boldsymbol{\alpha}$*-curves and* $\boldsymbol{\beta}$*-curves which are disjoint from the basepoints;*
(ii) *Handleslides among* $\boldsymbol{\alpha}$*-curves (resp.* $\boldsymbol{\beta}$*-curves) which keep away from the basepoints;*
(iii) *(De)stabilizations.*

Definition 2.5. For a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$, let $D_1, D_2, \cdots, D_h$ denote the closures of the components of $\Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. A *domain* is of the form $D = \sum_{i=1}^{h} a_i D_i$ with $a_i \in \mathbb{Z}$. D is a *positive domain* if $a_i \geq 0$ for $1 \leq i \leq h$. D is a *periodic domain* if ∂D is a sum of $\boldsymbol{\alpha}$-curves and $\boldsymbol{\beta}$-curves and $n_{\boldsymbol{z}}(D) = \{0\}$. Here we use $n_p(D)$ to denote the local multiplicity of D at $p \in \Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ and $n_{\boldsymbol{z}}(D) = \{n_{z_1}(D), n_{z_2}(D), \cdots, n_{z_m}(D)\}$.

Definition 2.6. A Heegaard diagram is said to be *admissible* if every non-trivial periodic domain has both positive and negative local coefficients at the Heegaard surface.

The following Proposition is essentially a corollary of [2, Corollary 3.12]. Here we prove it in a different way.

Proposition 2.7. *If* $H_1(M, \mathbb{Z}) = 0$ *and the balanced bipartite spatial graph* G_{V_1,V_2} *is connected, then there is no non-trivial periodic domain on the Heegaard diagram. Therefore any Heegaard diagram of* G_{V_1,V_2} *is admissible.*

Proof. Suppose the closures of the components of $\Sigma \backslash \boldsymbol{\alpha}$ are $\{A_1, A_2, \cdots, A_{d-g+1}\}$, and the closures of the components of $\Sigma \backslash \boldsymbol{\beta}$ are $\{B_1, B_2, \cdots, B_{d-g+1}\}$. Then by [7, Sequence (4)] any periodic domain has the form

$$P = \sum_{i=1}^{d-g+1} (a_i A_i + b_i B_i),$$

for $a_i, b_i \in \mathbb{Z}$. The pieces A_i and B_i correspond to vertices in V_1 and V_2 respectively, and basepoints correspond to edges in E. Choose a spanning tree for G, which is always a path passing through all vertices of G. Orient the path so that it starts from a vertex in V_1 and ends at a vertex in V_2. Label the elements in V_1, V_2 and E so that when we travel along the path we meet $v_1 \in V_1, u_1 \in V_2, v_2 \in V_1, u_2 \in V_2, \cdots, v_{d-g+1} \in$

$V_1, u_{d-g+1} \in V_2$ and cross $e_1, e_2, \cdots, e_{2d-2g+1}$ with $e_i \in E$. In other words, we can relabel $\{A_1, A_2, \cdots, A_{d-g+1}\}$, $\{B_1, B_2, \cdots, B_{d-g+1}\}$ and $\boldsymbol{z}$ so that $z_{2i-1} \in A_i \cap B_i$ for $1 \leq i \leq d-g+1$ and $z_{2i} \in B_i \cap A_{i+1}$ for $1 \leq i \leq d-g$. By the condition that $n_{z_{2i-1}}(P) = 0$, we have $a_i + b_i = 0$ for $1 \leq i \leq d-g+1$, and by the condition that $n_{z_{2i}}(P) = 0$, we have $a_i + b_{i+1} = 0$ for $1 \leq i \leq d-g$. Therefore there exists a constant c so that $a_i = c$ and $b_i = -c$ for all $1 \leq i \leq d-g+1$. Then we have $P = \sum_{i=1}^{d-g+1}(a_i A_i + b_i B_i) = c(\sum_{i=1}^{d-g+1} A_i - \sum_{i=1}^{d-g+1} B_i) = c(\Sigma - \Sigma) = \emptyset$. $\square$

In particular when $M = S^3$, the Heegaard diagram constructed after Definition 2.3 is always admissible in the case that G_{V_1,V_2} is connected. But when it is not connected, the Heegaard diagram obtained is usually not admissible.

Later on, we will see that not all Heegaard diagrams can be used in the definition of Heegaard Floer homology. The Heegaard Floer complex is only well defined on an admissible Heegaard diagram. In fact, even though the condition of Proposition 2.7 is not satisfied, we can always make a Heegaard diagram admissible by isotopies of $\boldsymbol{\beta}$-curves supported in the complement of $\boldsymbol{z}$. The proof can be found in [7, Proposition 3.6] and [2, Proposition 3.15]. Moreover, two admissible Heegaard diagrams can be connected by Heegaard moves so that the intermediate Heegaard diagrams are all admissible. From now on, we only work with admissible Heegaard diagrams.

3. Heegaard Floer complex

Most terminologies and notations are inherited from [6]. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ be an admissible Heegaard diagram for the balanced bipartite spatial graph G_{V_1,V_2} in the closed 3-manifold M. Define

$$\begin{aligned} \mathrm{Sym}^d(\Sigma) &= \Sigma^{\times d}/S_d, \\ \mathbb{T}_{\boldsymbol{\alpha}} &= (\alpha_1 \times \alpha_2 \times \cdots \times \alpha_d)/S_d \\ \text{and } \mathbb{T}_{\boldsymbol{\beta}} &= (\beta_1 \times \beta_2 \times \cdots \times \beta_d)/S_d, \end{aligned}$$

where S_d is the symmetric group with d letters.

Following the method in [6], we can show that $\mathrm{Sym}^d(\Sigma)$ is a smooth manifold, and that a complex structure j on Σ naturally endows $\mathrm{Sym}^d(\Sigma)$ an almost complex structure $\mathrm{Sym}^d(j)$, with respect to which $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ are totally real submanifolds of $\mathrm{Sym}^d(\Sigma)$.

We can suppose that $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ are in general position, which means that they intersect transversely.

Definition 3.1. Let $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, a *Whitney disk* connecting x to y is a continuous map $u : \mathbb{D} \to \mathrm{Sym}^d(\Sigma)$ so that $u(-i) = x$, $u(i) = y$, $u(e_1) \subset \mathbb{T}_{\boldsymbol{\alpha}}$ and $u(e_2) \subset \mathbb{T}_{\boldsymbol{\beta}}$. Here $\mathbb{D}$ is the unit disk, $e_1 = \{z \in \partial(\mathbb{D}) | \mathrm{Re}(z) \geq 0\}$ and $e_2 = \{z \in \partial(\mathbb{D}) | \mathrm{Re}(z) \leq 0\}$. Let $\pi(x, y)$ be the set of homotopy classes of Whitney disks connecting x to y.

Given a $\phi \in \pi(x, y)$, let $D(\phi) := \Sigma_{i=1}^{h} n_{p_i}(\phi) D_i$ be the associated domain, where p_i is an arbitrary point in $\mathrm{int}(D_i)$ and $n_{p_i}(\phi)$ is the algebraic intersection number $\phi^{-1}(\{p_i\} \times \mathrm{Sym}^{d-1}(\Sigma))$. Let $\mathcal{M}(\phi)$ be the moduli space of pseudo-holomorphic representatives of $\phi \in \pi(x, y)$ and $\widehat{\mathcal{M}}(\phi) := \mathcal{M}(\phi)/\mathbb{R}$ be the unparametrized moduli space. Also, let $\mu(\phi)$ denote the Maslov index of ϕ. When $\mu(\phi) = 1$ the space $\widehat{\mathcal{M}}(\phi)$ is a compact zero-dimensional manifold ([2, Corollary 6.4] and [6, Theorem 3.18]).

Definition 3.2. Let $\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ be the vector space over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ generated by points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Also define a linear map

$$\begin{aligned} \partial : \mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}) &\rightarrow \mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}) \\ x &\mapsto \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1, n_{\boldsymbol{z}}(\phi)=\{0\}} \sharp\widehat{\mathcal{M}}(\phi) \cdot y, \end{aligned}$$

where $\sharp\widehat{\mathcal{M}}(\phi)$ is the number of elements in $\widehat{\mathcal{M}}(\phi)$ modulo 2. That the right-hand side has only finitely many non-trivial terms follows from the admissibility of the Heegaard diagram.

Theorem 3.3. (i) $(\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}), \partial)$ *is a chain complex. Namely* $\partial^2 = 0$.
(ii) *The homology of the chain complex above is a topological invariant of* G_{V_1,V_2}*, denoted by* $\mathrm{HFG}(S^3, G_{V_1,V_2})$

Proof. (i) This is analogous to [6, Theorem 4.1]. (ii) With Theorem 2.4 in hand, the proof follows from [6, Section 7∼ 11]. □

Remark 3.4. *When G is not connected, the splitting $V = V_1 \coprod V_2$ is not unique and the homology depends on the splitting. When G is connected, the choice of V_1 and V_2 is unique. In fact, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ is a Heegaard diagram for G_{V_1,V_2}, then $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{z})$ is a Heegaard diagram for G_{V_2,V_1}. There is a canonical chain isomorphism between* $\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ *and* $\mathrm{CFG}(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{z})$*. Therefore when G is connected, the homology only depends on the topology type of G.*

Define the set

$$S := \{(x_1, x_2, \cdots, x_d) | x_i \in \alpha_i \cap \beta_{\sigma(i)} \text{ for some } \sigma \in S_d\}.$$

Then it is easy to see that there is a canonical identification between $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and S. We do not distinguish these two sets sometimes. Given $x, y \in S$, choose a multi-path $a \subset \bigcup_{i=1}^d \alpha_i$ connecting x to y, and a multi-path $b \subset \bigcup_{i=1}^d \beta_i$ connecting y to x. Then $a + b$ is a one-cycle in Σ. The homology class $[a + b] \in H_1(M \backslash \nu(G), \mathbb{Z})$ does not depend on the choice of a and b, where $\nu(G)$ is the interior of a regular neighbourhood of $G \subset M$. This is because for different a' and b', the difference $(a + b) - (a' + b')$ is a union of some $\boldsymbol{\alpha}$-curves and $\boldsymbol{\beta}$-curves.

Definition 3.5. Given $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, choose a and b as above. Then we define a relative grading $A : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \rightarrow H_1(M \backslash \nu(G), \mathbb{Z})$ by the formula

$$A(x) - A(y) = [a + b] \in H_1(M \backslash \nu(G), \mathbb{Z}).$$

Lemma 3.6. *If $A(x) - A(y) \neq 0$, then the subset of elements in $\pi_2(x, y)$ with $n_{\boldsymbol{z}}(\phi) = \{0\}$ is empty.*

Proof. If there exists a $\phi \in \pi(x, y)$ with $n_{\boldsymbol{z}} = \{0\}$, then the associated domain $D(\phi)$ is an empty domain connecting x to y. Then $A(x) - A(y) = \partial D(\phi) = 0 \in H_1(M \backslash \nu(G), \mathbb{Z})$. □

We can define a relative $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ in the following way. As we said in Definition 2.3, the orientation of Σ is inherited from that of M. We orient the $\boldsymbol{\alpha}$-curves $\alpha_1, \alpha_2, \cdots, \alpha_d$ and the $\boldsymbol{\beta}$-curves $\beta_1, \beta_2, \cdots, \beta_d$ arbitrarily. Then $\mathrm{Sym}^d(\Sigma)$, $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ have the orientations which are induced from the product orientations of $\Sigma^{\times d}$, $\alpha_1 \times \alpha_2 \times \cdots \times \alpha_d$ and $\beta_1 \times \beta_2 \times \cdots \times \beta_d$, respectively. For a generator $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$,

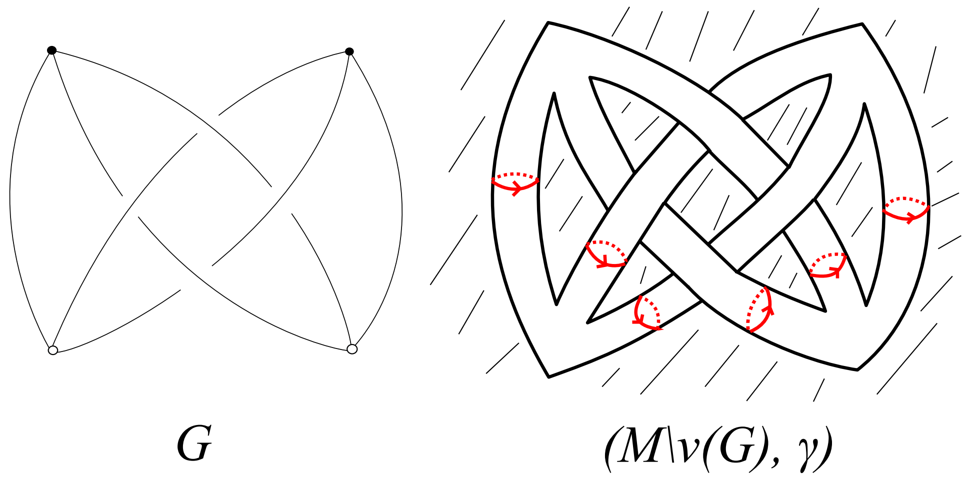


FIGURE 2.

the sign of the intersection gives its grading in $\mathbb{Z}/2\mathbb{Z} = \{+1, -1\}$, and we denote it by $\mathrm{sign}(x)$.

Identify $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $(x_1, x_2, \cdots, x_d) \in S$, where $x_i \in \alpha_1 \cap \beta_{\sigma(i)}$ for some $\sigma \in S_d$. Denote the sign of the intersection point x_i in Σ by $\mathrm{sign}(x_i)$. Then we have the relation $\mathrm{sign}(x) = \dfrac{d(d-1)}{2}\mathrm{sign}(\sigma)\prod_{i=1}^{d}\mathrm{sign}(x_i)$ (see [1, Lemma 2.12]).

We see that the differential in $\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ preserves the grading A and change the relative $\mathbb{Z}/2\mathbb{Z}$-grading. So we have the following splitting.

Theorem 3.7. $(\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}), \partial)$ *splits along* $H_1(S^3 \backslash \nu(G), \mathbb{Z})$. *Namely*

$$\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}) = \bigoplus_{i \in H_1(S^3 \backslash \nu(G), \mathbb{Z})} \mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}, i).$$

We have

$$\mathrm{HFG}_j(S^3, G_{V_1,V_2}) = \bigoplus_{i \in H_1(S^3 \backslash \nu(G), \mathbb{Z})} \mathrm{HFG}_j(S^3, G_{V_1,V_2}, i),$$

for $j = +1, -1$.

Remark 3.8. *The relative gradings do not depend on the choice of Heegaard diagrams.*

Definition 3.9. The *Euler characteristic* for $\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ is

$$\chi(\mathrm{CFG}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})) := \sum_{x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \mathrm{sign}(x) \cdot A(x).$$

4. PROPERTIES

Proposition 4.1. *For the balanced bipartite spatial graph* G_{V_1,V_2},

$$\mathrm{HFG}(M^3, G_{V_1,V_2}) \cong \mathrm{SFH}(M^3 \setminus \nu(G), \gamma),$$

where $\mathrm{SFH}(M^3 \setminus \nu(G), \gamma)$ *is the sutured Floer homology with the meridian anuuli of all edges as the sutures.*

Proof. The proof follows directly from the construction of these two homologies. $\square$

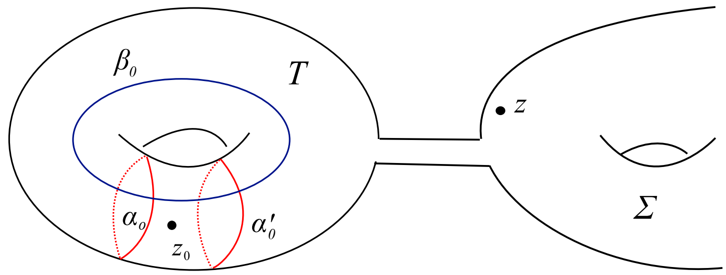


FIGURE 3.

Proposition 4.2. *For the balanced bipartite spatial graph G_{V_1,V_2}, if G_{V_1,V_2} has two vertices $v \in V_1$ and $u \in V_2$ connected by an edge $e \in E$ with valencies $d(v) = 1$ and $d(u) > 1$, then* $\mathrm{HFG}(M, G_{V_1,V_2}) = 0$.

Proof. Removing the edge e and the vertex v from G_{V_1,V_2}, we get a new graph G', which is not balanced anymore. But we can still consider its Heegaard diagram

$$H' = (\Sigma, \boldsymbol{\alpha'} = \{\alpha_1, \cdots, \alpha_{d-1}\}, \boldsymbol{\beta'} = \{\beta_1, \cdots, \beta_d\}, \boldsymbol{z'} = \{z_1, \cdots, z_{m-1}\}),$$

where $d - g + 1 = |V_2|$ and $m = |E|$, the number of edges in G_{V_1,V_2}. By applying [2, Proposition 3.15], we can show that H' is isotopic to an admissible Heegaard diagram. In fact, the essential condition needed in the proof is that elements of $\boldsymbol{\alpha'}$ (resp. $\boldsymbol{\beta'}$) are linearly independent in $H_1(\Sigma \setminus \boldsymbol{z'}, \mathbb{Z})$.

Therefore we can assume that H' itself is admissible. Suppose $z \in \boldsymbol{z}$ is a basepoint corresponding to an edge adjacent to u in G'. Then

$$\begin{aligned} H &= H' \sharp (T, \{\alpha_0, \alpha_0'\}, \{\beta_0\}, z_0) \\ &= (\Sigma \sharp T, \boldsymbol{\alpha} = \boldsymbol{\alpha'} \cup \{\alpha_0, \alpha_0'\}, \boldsymbol{\beta} = \boldsymbol{\beta'} \cup \{\beta_0\}, \boldsymbol{z} = \boldsymbol{z'} \cup \{z_0\}), \end{aligned}$$

where the connected sum takes place near z, is a Heegaard diagram for G_{V_1,V_2}. A periodic domain P for H is disjoint from z and z_0, so it must be supported in Σ. This means that P is a periodic domain for H', having both positive and negative local multiplicities. Therefore H is admissible as well. From H we see that $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} = \emptyset$, so $\mathrm{HFG}(M, G_{V_1,V_2}) = 0$. $\square$

Proposition 4.3. *Suppose G_{V_1,V_2} is a balanced bipartite spatial graph in the integral homology 3-sphere M, and I_{V_1,V_2} is a subgraph of G which has the same component number as G_{V_1,V_2}. Then there is a filtration so that* $\mathrm{HFG}(M, G_{V_1,V_2})$ *is the associated graded homology of* $\mathrm{HFG}(M, I_{V_1,V_2})$ *with respect to this filtration.*

Proof. We only need to consider the case that I is obtained from G by removing a non-separating edge e. Suppose now that $H_G = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$ is a Heegaard diagram of G. Then $H_I = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z} \setminus \{z_e\})$ is a Heegaard diagram for I, where z_e corresponds to the edge e. By using the method in the proof of [2, Proposition 3.15], we can make H_G admissible by isotopy moves of $\boldsymbol{\beta}$-curves. When we apply the method, we can only use the basepoints in $\boldsymbol{z} \setminus \{z_e\}$, while keeping the basepoint z_e away from the winding regions. In this way H_I becomes admissible simultaneously. We still use H_G and H_I to denote the admissible Heegaard diagrams after the isotopy moves.

Now $\mathrm{CFG}(H_G)$ and $\mathrm{CFG}(H_I)$ have the same generating set $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. By definition, given a generator $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, the differential of $\mathrm{CFG}(H_I)$ is

$$\begin{aligned} \partial^I(x) &= \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(x,y), \mu(\phi)=1, \\ n_{\boldsymbol{z}\setminus\{z_e\}}(\phi)=\{0\}}} \sharp\widehat{\mathcal{M}}(\phi) \cdot y \\ &= \sum_{i=0}^{\infty} \partial_i(y), \end{aligned}$$

where $\partial_i(x) = \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(x,y), \mu(\phi)=1, \\ n_{\boldsymbol{z}\setminus\{z_e\}}(\phi)=\{0\}, n_{z_e}(\phi)=i}} \sharp\widehat{\mathcal{M}}(\phi) \cdot y$. Here we use the fact that if $\phi \in \pi_2(x,y)$ has a pseudo-holomorphic representative, then $n_p(\phi) \geq 0$ for any $p \in \Sigma \setminus \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. Moreover, ∂_0 coincides with the differential ∂^G in $\mathrm{CFG}(H_G)$.

Since M is an integral homology 3-sphere, we have $H_1(M \backslash \nu(G), \mathbb{Z}) = \mathbb{Z}^{g(G)} = \mathbb{Z}^{g(I)} \oplus \mathbb{Z}\langle m_e \rangle$, where $\mathbb{Z}\langle m_e \rangle$ is a summand generated by the meridian m_e of e whose orientation is induced from that of Σ. (The boundary of a disk neighbourhood of z_e on Σ is m_e.) Here $g(G) = \sum_{i=1}^{k} g(G_i)$ with $g(G_i)$ the genus of $\overline{\nu(G_i)}$. For $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, their grading difference in $\mathrm{CFG}(H_G)$ is

$$A^G(x) - A^G(y) = [a+b] \in H_1(M \backslash \nu(G), \mathbb{Z}) = \mathbb{Z}^{g(I)} \oplus \mathbb{Z}\langle m_e \rangle,$$

where a and b are the muti-paths in Σ connecting x to y along $\boldsymbol{\alpha}$ and y to x along $\boldsymbol{\beta}$, respectively. The grading difference in $\mathrm{CFG}(H_I)$ is

$$A^I(x) - A^I(y) = [a+b] \in \mathbb{Z}^{g(I)}.$$

In fact, $A^I(x) - A^I(y)$ is the projection of $A^G(x) - A^G(y)$ onto $\mathbb{Z}^{g(I)}$. We define a new grading $B : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathbb{Z}$ on $\mathrm{CFG}(H_I)$ by requiring that $B(x) - B(y)$ is the projection of $A^G(x) - A^G(y)$ onto $\mathbb{Z}\langle m_e \rangle$. Then if $\partial_i(x) \neq 0$, $B(x) - B(\partial_i(x)) = i$ for $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $i \geq 0$. With respect to the grading B, the associated graded homology of $(\mathrm{CFG}(H_I), \partial_I)$ is $\mathrm{HFG}(M, G_{V_1,V_2})$.

□

4.1. **Euler characteristic.** Friedl, Juhász and Rasmussen in [1] defined a torsion invariant for sutured manifolds and showed that it is the Euler characteristic of the sutured Floer homology. We see from Proposition 4.1 that the homology we defined in this paper is a special case of sutured Floer homology. By referring the construction in [1], we give a combinatorial interpretation of the Euler characteristic of the homology in the case that $M = S^3$ and G is connected. Precisely, we show how to define it from a graph projection on the 2-sphere S^2.

In this paper, we only construct it for the case that $|V_1| = |V_2| = 1$ (e.g. the θ graph). Let $G = (\{u, v\}, E = \{e_1, e_2, \cdots, e_m\})$ be such a spatial graph, and D be one of its graph projections on S^2. The projection D divides S^2 into several regions. Mark the regions adjoining u by $*$. Supposing there are n crossings $\{c_1, c_2, \cdots, c_n\}$ on D, then it is easy to see that there are n regions $R_1, R_2, \cdots, R_n$ which are not marked by $*$ (see Figure 6). Each crossing point adjoins four regions. A *state* of D is a bijection $s : \{c_1, c_2, \cdots, c_n\} \to \{R_1, R_2, \cdots, R_n\}$ so that each crossing is mapped to a region adjacent to it. We use black dots in the corners to indicate the bijection. See Figure 4 for an example.

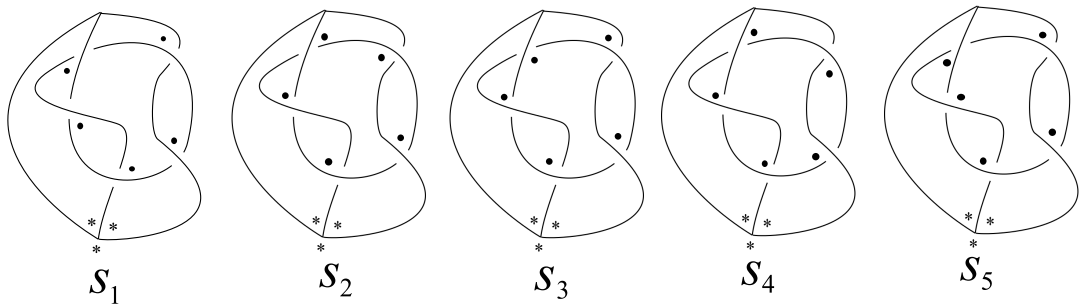


FIGURE 4. There are five states for this projection.

Orient each edge so that it directs from u to v, and assign an variable t_i to each edge e_i for $1 \leq i \leq m$. For each state s, define $\mathrm{sign}(s)$ to be the sign of the permutation of the subindices of $\{R_1, R_2, \cdots, R_n\}$ induced by s, and let $m(s) := \prod_{i=1}^{n} m_i^{s(c_i)}$, where $m_i^{s(c_i)}$ is defined by the rule in Figure 5.

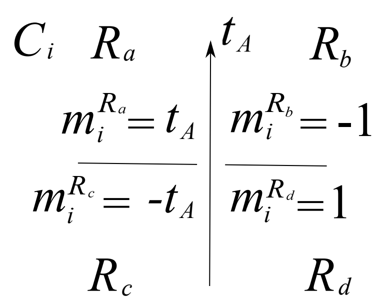


FIGURE 5.

We define the Laurent polynomial $\tau(t_1, t_2, \cdots, t_m) := \sum_s \mathrm{sign}(s) m(s)$. Then

$$\tau(t_1, t_2, \cdots, t_m) \in \mathbb{Z}H_1(S^3 \setminus \nu(G), \mathbb{Z}),$$

where $H_1(S^3 \setminus \nu(G), \mathbb{Z}) \cong \langle t_1, t_2, \cdots, t_m | t_1 t_2 \cdots t_m = 1 \rangle$.

Theorem 4.4. *Up to an overall multiplication of* $\pm t_1^{\pm}, \pm t_2^{\pm}, \cdots, \pm t_m^{\pm}$*, the polynomial* $\tau(t_1, t_2, \cdots, t_m)$ *coincides with the Euler characteristic of* $\mathrm{HFG}(S^3, G)$*.*

Example 4.5. *For the graph projection in Figure 4, we see there are five states. State* s_1 *sends* $(c_1, c_2, c_3, c_4, c_5)$ *to* $(R_2, R_3, R_4, R_1, R_5)$*, so it has* $\mathrm{sign}(s_1) = -1$*. Similarly, we see that* $\mathrm{sign}(s_2) = \mathrm{sign}(s_5) = 1$ *and* $\mathrm{sign}(s_3) = \mathrm{sign}(s_4) = -1$*. By the rule in Figure 5, we calculate* $m(s_i)$ *for* $1 \leq i \leq 5$*. We get* $m(s_1) = t_2^3$*,* $m(s_2) = 1$*,* $m(s_3) = t_2$*,* $m(s_4) = t_3$ *and* $m(s_5) = t_2^2$*. Therefore* $\tau(t_1, t_2, t_3) = -t_2^3 + 1 - t_2 - t_3 + t_2^2$*.*

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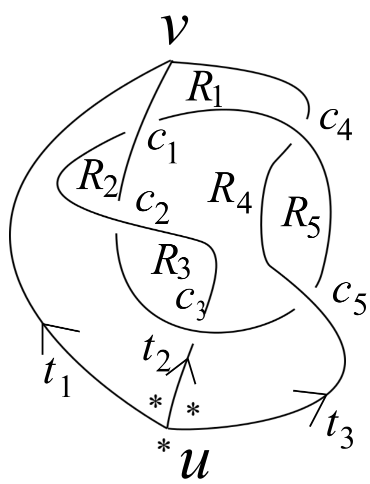

FIGURE 6.

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INTERDISCIPLINARY CENTER FOR MATHEMATICAL SCIENCES, GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA, TOKYO 153-8914, JAPAN

INSTITUTE FOR BIOLOGY AND MATHEMATICS OF DYNAMICAL CELL PROCESSES (IBMATH), UNIVERSITY OF TOKYO, 3-8-1 KOMABA, TOKYO 153-8914, JAPAN
E-mail address: `bao@ms.u-tokyo.ac.jp`