On the D-affinity of the flag variety in type B_2 *

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Abstract

The flag varieties in characteristic 0 are well-known to be D-affine. In positive characteristic, however, only those in type A_1 and A_2 have been proved to be so. In this paper we will show in type B_2 the cohomology vanishing of the first term in the p-filtration of the sheaf of differential operators on the flag variety. This is a necessary condition for the variety to be D-affine.

Let $\mathfrak{X}$ be a smooth variety over an algebraically closed field k, and let $\mathcal{D}_{\mathfrak{X}}$ be the sheaf of differential operators on $\mathfrak{X}$. Then $\mathfrak{X}$ is said to be D-affine iff the following two conditions hold: (i) for any $\mathcal{D}_{\mathfrak{X}}$-module $\mathcal{M}$ quasi-coherent over $\mathcal{O}_{\mathfrak{X}}$ the natural morphism $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}(\mathfrak{X})} \mathcal{M}(\mathfrak{X}) \to \mathcal{M}$ is epic, (ii) $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) = 0$ for $i > 0$.

In characteristic 0 the flag variety for a semisimple algebraic group is known to be D-affine [BB]. This is one of the the keys to the celebrated proofs by Brylinski and Kashiwara [BK] and Beilinson and Bernstein [BB] of the Kazhdan-Lusztig conjecture [KL] on the irreducible characters for finite dimensional semisimple k-Lie algebras.

In positive characteristic B. Haastert [H] has proved that the projective space $\mathbb{P}^n_k$ is D-affine, and that when $\mathfrak{X}$ is the flag variety G/B with G a simply connected simple algebraic group over k and B a Borel subgroup, any $\mathcal{D}_{\mathfrak{X}}$-module quasi-coherent over $\mathcal{O}_{\mathfrak{X}}$ is generated by the global sections even over $\mathcal{O}_{\mathfrak{X}}$. He has also verified the condition (ii) for G of type A_2. If p is the positive characteristic, $\mathcal{D}_{\mathfrak{X}}$ admits a filtration $(\mathcal{D}_r)$, called the p-filtration. If G_r is the r-th Frobenius kernel of G and if $-\rho$ is half sum of the roots of B, Haastert identifies $\mathcal{D}_r$ with the sheaf $\mathcal{L}(\mathrm{ind}_B^{G_rB}(2(p^r-1)\rho))$ induced by the B-module $\mathrm{ind}_B^{G_rB}(2(p^r-1)\rho)$. For type A_2 he checks that all the G_rB-composition

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factors of $\mathrm{ind}_B^{G_rB}(2(p^r-1)\rho)$ have dominant highest weights, hence (ii) follows in this case from Kempf's vanishing theorem. In type B_2, however, not all composition factors of $\mathrm{ind}_B^{G_rB}(2(p^r-1)\rho)$ have dominant highest weights. We will nevertheless show in this note for the first term $\mathcal{D}_1$ of the p-filtration

Theorem. *If G is of type B_2,*

$$\mathrm{H}^i(G/B, \mathcal{D}_1) = 0 \quad \textit{for } i > 0.$$

According to N. Lauritzen (private communication, see §1 below) for any variety $\mathfrak{X}$ admitting a Frobenius splitting the condition (ii) is equivalent to the vanishing of all higher cohomologies of all $\mathcal{D}_r$, $r \in \mathbb{N}^+$. The flag variety is Frobenius split by [MR] (cf. also [K95]). Thus our result is a necessary condition for the flag variety in type B_2 to be D-affine.

The present work was partly inspired by the announcement of Xi [X] in [X99]. We are grateful to N. Lauritzen for allowing us to include his unpublished observation. The second author also thanks R. Bøgvad for a helpful discussion on Lauritzen's assertion. The first author would like to thank the Department of Mathematics, Osaka City University for a very pleasant stay there during the month of November 1999.

1° p-filtrations

(1.1) Let $\mathfrak{X}$ be a smooth variety over an algebraically closed field of characteristic $p > 0$. If $\mathcal{O}_{\mathfrak{X}}^{(r)}$ is the sheaf on $\mathfrak{X}$ defined by $\mathcal{O}_{\mathfrak{X}}^{(r)}(V) = \{a^{p^r} \mid a \in \mathcal{O}_{\mathfrak{X}}(V)\}$ for each open subset V of $\mathfrak{X}$ and if $\mathcal{D}_r = \mathcal{D}_{\mathfrak{X},r} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$, then $(\mathcal{D}_r)_{r\in\mathbb{N}}$ defines a filtration of $\mathcal{D}_{\mathfrak{X}}$, called the p-filtration of $\mathcal{D}_{\mathfrak{X}}$. Recall that $\mathfrak{X}$ is said to be Frobenius split iff $\mathcal{O}_{\mathfrak{X}}^{(1)}$ is a direct summand of $\mathcal{O}_{\mathfrak{X}}$ as $\mathcal{O}_{\mathfrak{X}}^{(1)}$-module.

Lemma (N. Lauritzen). *Assume $\mathfrak{X}$ is Frobenius split. If $r < s$, then $\mathcal{D}_r$ is a direct summand of $\mathcal{D}_s$ as sheaf of abelian groups.*

Proof: By the hypothesis $\mathcal{O}_{\mathfrak{X}}^{(s-r)}$ is a direct summand of $\mathcal{O}_{\mathfrak{X}}$ as $\mathcal{O}_{\mathfrak{X}}^{(s-r)}$-modules, hence $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}^{(s)}}(\mathcal{O}_{\mathfrak{X}}^{(s-r)}, \mathcal{O}_{\mathfrak{X}}^{(s-r)})$ is a direct summand of $\mathcal{D}_s$ as $\mathcal{O}_{\mathfrak{X}}^{(s)}$-modules. As the morphism $F^{s-r} : \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}^{(s-r)}$ via $a \mapsto a^{p^{s-r}}$ is invertible, there is an isomorphism of sheaves of rings $\mathcal{D}_r \to \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}^{(s)}}(\mathcal{O}_{\mathfrak{X}}^{(s-r)}, \mathcal{O}_{\mathfrak{X}}^{(s-r)})$ via $\delta \mapsto F^{s-r} \circ \delta \circ F^{-(s-r)}$, hence the assertion.

(1.2) **Proposition.** *Assume $\mathfrak{X}$ is Frobenius split. Then for each $i \in \mathbb{N}$*

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) = 0 \quad \textit{iff} \quad \mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r) = 0 \quad \forall r \in \mathbb{N}.$$

Proof: As $\mathfrak{X}$ is noetherian, $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) \simeq \varinjlim_r \mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r)$, hence "if" is clear. Assume $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) = 0$. If $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r) \neq 0$ for some r, any $\delta \in \mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r) \setminus 0$ must vanish in some

$\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_s)$, $s > r$. But that would contradict the above lemma that $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r)$ should be a direct summand of $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_s)$.

2° **Type** B_2

From now on throughout the rest of the paper k will denote an algebraically closed field of positive characteristic p, and $\mathfrak{X}$ the flag variety G/B with G a simply connected simple algebraic group over k of type B_2 and B a Borel subgroup of G. Let T be a maximal torus of B. We choose the roots of B to be negative, and denote the simple roots by α_1, α_2 with α_1 short. Let ω_1 and ω_2 be the fundamental weights of T such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

(2.1) Let G_1 (resp. B_1) be the Frobenius kernel of G (resp. B), and let $\hat{Z} = \mathrm{ind}_B^{G_1 B}$ (resp. $\tilde{Z} = \mathrm{ind}_{B_1 T}^{G_1 T}$) be the induction functor from the category of B-modules (resp. B_1T-modules) to the category of G_1B-modules (resp. G_1T-modules). Composing with the forgetful functor, $\hat{Z}$ coincides with $\tilde{Z}$ [J, II.9.1]. Let $\mathrm{H}^0 = \mathrm{ind}_B^G$ (resp. $\mathrm{H}^0(\alpha_1, ?)$) be the induction functor from the category of B-modules to G-modules (resp. $P(\alpha_1)$-modules, $P(\alpha_1)$ being the minimal parabolic subgroup of G containing B associated with α_1). We will abbreviate the right derived functors $\mathrm{R}^\bullet\mathrm{H}^0$ of H^0 as $\mathrm{H}^\bullet$. By Haastert's identification [H, 4.3.3] we have to show

$$\mathrm{H}^i(\hat{Z}(2(p-1)\rho)) = 0 \quad \forall i > 0. \tag{1}$$

We will denote the G_1T-socle series of $\tilde{Z}(2(p-1)\rho)$ by soc^j, $j \in \mathbb{N}^+$, and its j-th socle layer $\mathrm{soc}^j/\mathrm{soc}^{j-1}$ by soc_j. As G_1 is normal in G, the G_1T-socle series coincides with the G_1-socle series [J, I.6.15, II.3.15], and hence each soc^j is G_1B-stable. Thus to see (1), it is enough to show $\mathrm{H}^i(\mathrm{soc}_j) = 0$ for all $i > 0$ and $j \in \mathbb{N}^+$.

Let X be the character group of T, and $\mathbb{Z}[X]$ be the group ring of X with the natural basis $e(\nu)$. By [J79, 5.3] the formal character of $\tilde{Z}(2(p-1)\rho)$ is given by

$$\begin{aligned} \mathrm{ch}\tilde{Z}(2(p-1)\rho) &= e(0) + e(2p\omega_1) + \chi(2\omega_1 + (p-3)\omega_2)e(2p\omega_1) \\ &+ \chi((p-4)\omega_1)e(3p\omega_1) + \chi((p-4)\omega_1)e(p\omega_1) \\ &+ \chi((p-2)\omega_1 + \omega_2)e(p\omega_1) + \chi((p-3)\omega_2)e(2p\omega_2) \\ &+ \chi((p-3)\omega_2)e(p\omega_2) + \chi(2\omega_1 + (p-2)\omega_2)e(p\omega_2) \\ &+ \chi((p-4)\omega_1 + \omega_2)e(p(-\omega_1 + 2\omega_2)) + \chi((p-4)\omega_1 + \omega_2)e(p\rho) \\ &+ \chi((p-2)\rho)e(p\rho), \end{aligned} \tag{2}$$

where $\chi = \sum_{i \geq 0} (-1)^i \mathrm{ch}\mathrm{H}^i$.

We will denote the simple G-module of highest weight λ by $L(\lambda)$. Recall that the simple G_1B (and G_1T) -modules have the form $L(\lambda) \otimes p\mu$ with $\lambda \in X_1$ and $\mu \in X$. Here X_1 denotes the set of restricted weights, i.e.

$$X_1 = \{r_1\omega_1 + r_2\omega_2 | 0 \leq r_1, r_2 < p\}.$$

(2.2) Assume first $p = 2$. In this case $\tilde{Z}(2\rho)$ is of dimension 8, yielding to direct computations. We first find the G_1T-socle layers of $\tilde{Z}(2\rho)$ to be

$$\begin{aligned} &\mathrm{soc}_1 = 2\rho, && \mathrm{soc}_2 = 4\omega_1 \oplus 2\omega_2, \\ &\mathrm{soc}_3 = L(\omega_2) \otimes 2\omega_2 \oplus 2\omega_1, && \mathrm{soc}_4 = L(\omega_2) \otimes 2\omega_1 \oplus 2\omega_2, \\ &\mathrm{soc}_5 = 2\omega_1 \oplus 2(-\omega_1 + \omega_2), && \mathrm{soc}_6 = k. \end{aligned}$$

To work that out, it is convenient to identify $\tilde{Z}(2\rho)$ with $\mathrm{coind}_{B_1T}^{G_1T}(k) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1)} k$ [J, II.9.1], where $\mathrm{Dist}(G_1)$ (resp. $\mathrm{Dist}(B_1)$) is the algebra of distributions on G_1 (resp. B_1). If U_1^+ is the Frobenius kernel of U^+ and if $\mathrm{Dist}(U_1^+)$ is the algebra of disributions on U_1^+, then $\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1)} k$ is isomorphic as k-linear spaces to $\mathrm{Dist}(U_1^+)$. Using the standard basis of $\mathrm{Dist}(G_1)$, one can explicitly compute the $\mathrm{Dist}(G_1)$-module structure of $\mathrm{coind}_{B_1T}^{G_1T}(k)$ to obtain the G_1T-socle layers of $\tilde{Z}(2\rho)$.

Thus by the tensor identity and by Kempf's vanishing theorem the only problem is to show $\mathrm{H}^i(\mathrm{soc}_5) = 0$ for all $i > 0$. But there is an exact sequence of B-modules

$$0 \longrightarrow \mathrm{soc}_5 \longrightarrow \mathrm{H}^0(\alpha_1, 2\omega_1) \xrightarrow{\pi} \mathrm{H}^0(\alpha_1, \omega_2) \longrightarrow 0.$$

As $\mathrm{H}^i(\mathrm{H}^0(\alpha_1, 2\omega_1)) \simeq \mathrm{H}^i(2\omega_1) = 0 = \mathrm{H}^i(\mathrm{H}^0(\alpha_1, \omega_2))$ for all $i > 0$, we have only to show that $\mathrm{H}^0(\pi)$ is surjective.

Note first that $\mathrm{Hom}_G(\mathrm{H}^0(2\omega_1), \mathrm{H}^0(\omega_2)) = k$ [J, II.6.24] and that $\mathrm{H}^0(\pi) \neq 0$. The latter follows from the commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^0(2\omega_1) & \xrightarrow{\sim} & \mathrm{H}^0(\mathrm{H}^0(\alpha_1, 2\omega_1)) & \xrightarrow{\mathrm{H}^0(\pi)} & \mathrm{H}^0(\mathrm{H}^0(\alpha_1, \omega_2)) & \xrightarrow{\sim} & \mathrm{H}^0(\omega_2) \\ & \searrow^{\mathrm{res}} & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & \swarrow^{\mathrm{res}} & \\ & & \mathrm{H}^0(\alpha_1, 2\omega_1) & \xrightarrow[\pi]{} & \mathrm{H}^0(\alpha_1, \omega_2), & & \end{array}$$

where the restriction maps are both surjective [J, II.14.15] (cf. also [K95]).

Dually, consider the homomorphism of Weyl modules $\Delta(\omega_2) \to \Delta(2\omega_1)$. Let $\mathbf{G_a}$ be the 1-dimensional unipotent group, u_1 and $u_2 : \mathbf{G_a} \to G$ be the morphisms defining the root subgroups $U_{-\alpha_1}$ and $U_{-\alpha_2}$, respectively, and let $F_i = (\mathrm{d}u_i)(1)$, $i = 1, 2$. If v^+ is the highest weight vector of $\Delta(2\omega_1)$, we may assume that the image of a highest weight vector of $\Delta(\omega_2)$ in $\Delta(2\omega_1)$ is F_1v^+. As the weight 0 appears in $\Delta(2\omega_1)$ with multiplicity 2 and as $F_2v^+ = 0$, we must have $F_1F_2F_1v^+ \neq 0$. On the other hand, the weight 0 appears in $\Delta(\omega_2)$ with multiplicity 1 and $\mathrm{soc}_G\Delta(\omega_2) = k$. It follows that the homomorphism $\Delta(\omega_2) \to \Delta(2\omega_1)$ is injective, and hence $\mathrm{H}^0(\pi)$ is surjective.

Remarks. (i) As an alternative to the above proof of the surjectivity of $\mathrm{H}^0(\pi)$ one may use the idea employed in the generic case below. What is required in the case at hand is the vanishing of $H^2(s_1.2\omega_1)$, where $s_1.2\omega_1 = s_1(2\omega_1 + \rho) - \rho$. The $p = 2$ case is not covered in [A81] but the methods there easily gives this particular vanishing result.

(ii) The B-module $\hat{Z}(2\rho)$ does not admit an excellent filtration of Polo [P]. Otherwise van der Kallen's height-length filtration [vdK] would be one, forcing soc_5 above to be isomorphic with $\mathrm{H}^0(\mathfrak{X}(w), 2\omega_1)$ for some Schubert scheme $\mathfrak{X}(w)$, $w \in W$, that is absurd.

(2.3) If $p = 3$, then (2.1.2) shows that all G_1T-composition factors of $\tilde{Z}(4\rho)$ have dominant highest weights. Hence

$$\mathrm{H}^i(\tilde{Z}(4\rho)) = 0 \quad \forall i > 0$$

by Kempf's vanishing theorem, as desired.

(2.4) Assume finally $p \geq 5$. In this case the Lusztig conjecture [L] on the irreducible characters for G-modules holds, and hence also the conjecture on the irreducible characters for G_1T-modules by direct computations using Jantzen's formula (2.1.2) or by [K89, 4.5 and 4.15]. Then we know from [AK] the G_1T-socle series of $\tilde{Z}(2(p-1)\rho)$:

$$\begin{aligned}
\mathrm{soc}_1 &= L((p-2)\rho) \otimes p\rho, \\
\mathrm{soc}_2 &= L((p-4)\omega_1) \otimes p\omega_1 \oplus L((p-4)\omega_1) \otimes p(\rho - \alpha_1) \\
&\quad \oplus L((p-4)\omega_1) \otimes p\rho \oplus L((p-4)\omega_1) \otimes 3p\omega_1 \\
&\quad \oplus L((p-3)\omega_2) \otimes p\omega_2 \oplus L((p-3)\omega_2) \otimes 2p\omega_1 \\
&\quad \oplus L((p-3)\omega_2) \otimes 2p\omega_2 \oplus L((p-2)\omega_1 + \omega_2) \otimes p\rho, \\
\mathrm{soc}_3 &= L((p-4)\omega_1 + \omega_2) \otimes p\rho \oplus L((p-4)\omega_1 + \omega_2) \otimes p(\rho - \alpha_1) \\
&\quad \oplus L((p-4)\omega_1 + \omega_2) \otimes p\omega_1 \oplus p\omega_2 \oplus 2p\omega_2 \oplus 2p\omega_1 \\
&\quad \oplus L(2\omega_1 + (p-3)\omega_2) \otimes p\omega_2 \oplus L(2\omega_1 + (p-3)\omega_2) \otimes 2p\omega_1, \\
\mathrm{soc}_4 &= L(2\omega_1 + (p-2)\omega_2) \otimes p\omega_2 \oplus L((p-2)\omega_1 + \omega_2) \otimes p\omega_1, \\
\mathrm{soc}_5 &= k.
\end{aligned}$$

Note that soc_2 and soc_3 contain nondominant composition factors. We shall check that even so we still have $\mathrm{H}^i(\mathrm{soc}_j) = 0$ for $i > 0$ also for $j = 2, 3$.

Consider first soc_2. We have an isomorphism of G_1B-modules

$$\mathrm{soc}_2 \simeq \coprod_{\lambda \in X_1} L(\lambda) \otimes \mathrm{Hom}_{G_1}(L(\lambda), \mathrm{soc}_2).$$

Hence we have only to examine the $L((p-4)\omega_1)$-isotypic component $L((p-4)\omega_1) \otimes \mathrm{Hom}_{G_1}(L((p-4)\omega_1), \mathrm{soc}_2)$. Let Q_1 be the G_1B-submodule of soc^2 containing soc^1 such that $Q_1/\mathrm{soc}^1 \simeq L((p-4)\omega_1) \otimes \mathrm{Hom}_{G_1}(L((p-4)\omega_1), \mathrm{soc}_2)$.

The weights of $\mathrm{Hom}_{G_1}(L((p-4)\omega_1), \mathrm{soc}_2)$ are $p\omega_1$, $p(\rho-\alpha_1)$, $p\rho$, and $3p\omega_1$, all appearing multiplicity free. It follows that there are G_1B-submodules $Q_2 > Q_3 > \mathrm{soc}^1$ of Q_1 such that $Q_3/\mathrm{soc}^1 \simeq L((p-4)\omega_1) \otimes p\omega_1$ while that Q_2/Q_3 has the composition factors $L((p-4)\omega_1) \otimes p\rho$ and $L((p-4)\omega_1) \otimes p(\rho-\alpha_1)$. Thus $Q_2/Q_3 \simeq L((p-4)\omega_1) \otimes \mathrm{Hom}_{G_1}(L((p-4)\omega_1), Q_2/Q_3)$. If $Q_4 = \mathrm{Hom}_{G_1}(L((p-4)\omega_1), Q_2/Q_3)$, we are reduced to showing $\mathrm{H}^i(Q_4) = 0$ for all $i > 0$.

We claim that there is a nonsplit exact sequence of B-modules

$$(1) \qquad 0 \longrightarrow p\rho - \alpha_1 \longrightarrow Q_4 \longrightarrow p\rho \longrightarrow 0.$$

Just suppose the sequence split. Then $L((p-4)\omega_1) \otimes p\rho$ would be a G_1B-submodule of Q_2/Q_3. Consider the exact sequence of G-modules

$$\mathrm{ind}_{G_1B}^G(Q_2 \otimes -p\rho) \longrightarrow \mathrm{ind}_{G_1B}^G(Q_2/Q_3 \otimes -p\rho) \longrightarrow \mathrm{R}^1\mathrm{ind}_{G_1B}^G(Q_3 \otimes -p\rho)$$

induced by the obvious short exact sequence of G_1B-modules. We have

$$\mathrm{ind}_{G_1B}^G(Q_2 \otimes -p\rho) \subset \mathrm{ind}_{G_1B}^G(\hat{Z}((p-2)\rho)) \simeq \mathrm{H}^0((p-2)\rho),$$
$$\mathrm{ind}_{G_1B}^G(Q_2/Q_3 \otimes -p\rho) \supset \mathrm{ind}_{G_1B}^G(L((p-4)\omega_1)) \simeq L((p-4)\omega_1),$$

while the G-composition factors of $\mathrm{R}^1\mathrm{ind}_{G_1B}^G(Q_3 \otimes -p\rho)$ are among those of $\mathrm{R}^1\mathrm{ind}_{G_1B}^G(L((p-2)\rho)) = 0$ and of $\mathrm{R}^1\mathrm{ind}_{G_1B}^G(L((p-4)\omega_1) \otimes -p\omega_2) \simeq L((p-4)\omega_1) \otimes \mathrm{H}^1(-\omega_2)^{(1)} = 0$. But $L((p-4)\omega_1)$ is not a composition factor of $\mathrm{H}^0((p-2)\rho)$ and we have a contradiction.

Hence (1) holds and this means that Q_4 fits into the exact sequence of B-modules

$$0 \longrightarrow Q_4 \longrightarrow \mathrm{H}^0(\alpha_1, p\rho) \xrightarrow{\pi} \mathrm{H}^0(\alpha_1, p\rho - \alpha_1) \longrightarrow 0.$$

As in (2.2) we have to show $\mathrm{H}^0(\pi)$ is surjective. If $s_1 \in W$ is the reflection associated to α_1, considerations as in [A80]/[J, II.6.12] yields an exact sequence of G-modules

$$0 \to \mathrm{H}^0(p\rho - \alpha_1) \to \mathrm{H}^1(s_1.p\rho) \to \mathrm{H}^0(p\rho) \xrightarrow{\mathrm{H}^0(\pi)} \mathrm{H}^0(p\rho - \alpha_1) \to \mathrm{H}^2(s_1.p\rho) \to 0.$$

But $\mathrm{H}^2(s_1.p\rho) = 0$ by [A81, §4], as desired.

Finally, consider soc_3. In this case we have only to consider the $L((p-4)\omega_1 + \omega_2)$-isotypic component $L((p-4)\omega_1 + \omega_2) \otimes \mathrm{Hom}_{G_1}(L((p-4)\omega_1 + \omega_2), \mathrm{soc}_3)$. In analogy with (3.2) we let Q_5 be the G_1B-submodule of soc^3 containing soc^2 such that $Q_5/\mathrm{soc}^2 \simeq L((p-4)\omega_1 + \omega_2) \otimes \mathrm{Hom}_{G_1}(L((p-4)\omega_1 + \omega_2), \mathrm{soc}_3)$. Then there are G_1B-submodules $Q_6 \supset Q_7 \supset \mathrm{soc}^2$ of Q_5 such that $Q_7/\mathrm{soc}^2 \simeq L((p-4)\omega_1 + \omega_2) \otimes p\omega_1$ and that Q_6/Q_7 has the composition factors $L((p-4)\omega_1 + \omega_2) \otimes p\rho$ and $L((p-4)\omega_1 + \omega_2) \otimes p(\rho - \alpha_1)$. If $Q_8 = \mathrm{Hom}_{G_1}(L((p-4)\omega_1 + \omega_2), Q_6/Q_7)$, it is enough to check $\mathrm{H}^i(Q_8) = 0$ for all $i > 0$.

Again we find that the short exact sequence of B-modules

$$0 \longrightarrow p\rho - \alpha_1 \longrightarrow Q_8 \longrightarrow p\rho \longrightarrow 0$$

is nonsplit, and we finish the verification as for soc_2.

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