THEORY OF MULTI-FANS

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ABSTRACT. We introduce the notion of a multi-fan. It is a generalization of that of a fan in the theory of toric variety in algebraic geometry. Roughly speaking a toric variety is an algebraic variety with an action of algebraic torus of the same dimension as that of the variety, and a fan is a combinatorial object associated with the toric variety. Algebro-geometric properties of the toric variety can be described in terms of the associated fan. We develop a combinatorial theory of multi-fans and define "topological invariants" of a multi-fan. A smooth manifold with an action of a compact torus of half the dimension of the manifold and with some orientation data is called a torus manifold. We associate a multi-fan with a torus manifold, and apply the combinatorial theory to describe topological invariants of the torus manifold. A similar theory is also given for torus orbifolds. As a related subject a generalization of the Ehrhart polynomial concerning the number of lattice points in a convex polytope is discussed.

1. INTRODUCTION

The purpose of the present paper is to develop a theory of multi-fans which is an outgrowth of our study initiated in the work [23] on the topology of torus manifolds (the precise definition will be given later). A multi-fan is a combinatorial object generalizing the notion of a fan in algebraic geometry. Our theory is combinatorial by nature but it is built so as to keep a close connection with the topology of torus manifolds.

It is known that there is a one-to-one correspondence between toric varieties and fans. A toric variety is a normal complex algebraic variety of dimension n with a $(\mathbb{C}^*)^n$-action having a dense orbit. The dense orbit is unique and isomorphic to $(\mathbb{C}^*)^n$, and other orbits have smaller dimensions. The fan associated with the toric variety is a collection of cones in $\mathbb{R}^n$ with apex at the origin. To each orbit corresponds a cone of dimension equal to the codimension of the orbit. Thus the origin is the cone corresponding to the dense orbit, one-dimensional cones correspond to maximal singular orbits and so on. The important point is the fact that the original toric variety can be reconstructed from the associated fan, and algebro-geometric properties of the toric variety can be described in terms of combinatorial data of the associated fan.

If one restricts the action of $(\mathbb{C}^*)^n$ to the usual torus $T = (S^1)^n$, one can still find the fan, because the orbit types of the action of the total group $(\mathbb{C}^*)^n$ can be detected by the isotropy types of the action of the subgroup T. Take a circle subgroup S of T which appears as an isotropy subgroup of the action. Then each connected component of the closure of the set of those points whose isotropy subgroup equals S is a T-invariant submanifold of real codimension 2, and contains a unique $(\mathbb{C}^*)^n$ orbit of complex codimension 1. We shall call such a submanifold a characteristic submanifold. If $M_1, \dots, M_k$ are characteristic submanifolds such that $M_1 \cap \dots \cap M_k$ is non-empty, then the submanifold $M_1 \cap \dots \cap M_k$ contains a unique $(\mathbb{C}^*)^n$-orbit of complex codimension k. This suggests the following definition of torus manifolds and associated multi-fans.

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Let M be an oriented closed manifold of dimension $2n$ with an effective action of n dimensional torus T with non-empty fixed point set M^T. A closed, connected, codimension two submanifold of M will be called characteristic if it is a connected component of the fixed point set of a certain circle subgroup S of T, and if it contains at least one T-fixed point. The manifold M together with a prefered orientation of each characteristic submanifold will be called a torus manifold. The multi-fan associated with the torus manifold M consists of cones in the Lie algebra $L(T)$ of T, with apex at the origin. If M_i is a characteristic submanifold and S_i is the circle subgroup of T which pointwise fixes M_i, then S_i together with the orientation of M_i determines an element v_i of $\mathrm{Hom}(S^1, T)$, and hence a one dimensional cone in the vector space $\mathrm{Hom}(S^1, T) \otimes \mathbb{R}$ canonically identified with $L(T)$. If $M_{i_i}, \dots, M_{i_k}$ are characteristic submanifolds such that their intersection contains at least one T-fixed point, and if $v_{i_1}, \dots, v_{i_k}$ are the corresponding elements in $\mathrm{Hom}(S^1, T)$, then the k-dimensional cone spanned by $v_{i_1}, \dots, v_{i_k}$ lies in the multi-fan associated with M. It should be noted that the intersection of characteristic submanifolds may not be connected in contrast with the case of toric manifolds where the intersection is always connected. For example, the intersection of a family of n characteristic submanifolds is a finite set consisiting of T-fixed points. These data are also incorporated in the definition of the associated multi-fan.

One of the differences between a fan and a multi-fan is that, while cones in a fan intersect only at their faces and their union covers the space $L(T)$ just once without overlap for complete toric varieties, it happens that the union of cones in a multi-fan covers $L(T)$ with overlap for torus manifolds. Also the same multi-fan corresponds to different torus manifolds. Nevertheless it turns out that important topological invariants of a torus manifold can be described in terms of the associated multi-fan. In fact it is furthermore possible to develop an abstract theory of multi-fans and to define various "topological" invariants of a multi-fan in such a way that, when the fan is associated with a torus manifold, they coincide with the ordinary topological invariants of the manifold. For example, the "multiplicity of overlap", which we call the degree of the multi-fan, equals the Todd genus for a unitary torus manifold (unitary toric manifold in the terminology in [23]).

Another feature of the theory of toric varieties is the correspondence between ample line bundles over a complete toric variety and convex polytopes. From a topological point of view this can be explained in the following way. Let (M, ω) be a compact symplectic manifold with a Hamiltonian T-action, and let $\Psi : M \to L(T)^*$ be an associated moment map. Then it is well-known ([1], [13]) that the image P of Ψ is a convex polytope. Moreover, if the de Rham cohomology class of ω is an integral class, then the polytope P is a lattice polytope up to translations in $L(T)^*$ identified with $\mathbb{R}^n$. Delzant [7] showed that the original symplectic manifold (M, ω) is equivariantly symplectomorphic to a complete non-singular toric variety and the form ω is transfomed into the first Chern form of an ample line bundle L over the toric variety. It is known that the number of lattice points in P is equal to the Riemann-Roch number

$$\int_M e^{c_1(L)} \mathcal{T}(M)$$

where $\mathcal{T}$ is the Todd class of M, see e.g. [9]. This sort of phenomenon was generalized to "presymplectic" toric manifolds by Karshon and Tolman [18], then to Spin^c toric manifolds by Grossberg and Karshon [10] and also to unitary toric manifolds by the second-named author [23] in the form which relates the equivariant index of the line bundle L regarded as an element of $K_T(M)$ to the Duistermaat-Heckman measure of the moment map associated with L. In these extended cases the form ω may be degenerate or the line bundle may not be ample, and consequently the image of the moment map may not be convex any longer. This leads us to consider more general figures which we call multi-polytopes. A multi-polytope is a pair of a multi-fan and an arrangement of affine hyperplanes in $L(T)^*$. A similar notion was introduced by Karshon and Tolman [18] and also by Khovanskii and Pukhlikov [21] for ordinary fans under the name twisted polytope and virtual polytope respectively. We shall develop a combinatorial theory of multi-polytopes as well; we define the Duistermaat-Heckman measure and the equivariant index in purely combinatorial fashion for multi-fans and multi-polytopes, and generalize above results in the combinatorial context. Also we shall introduce a combinatorial counterpart of moment map which can be used to interpret the combinatorial Duistermaat-Heckman measure.

In carrying out the above program, the use of equivariant homology and cohomology plays an important role. First note that the group $\mathrm{Hom}(S^1, T)$ can be canonically identified with the equivariant integral homology group $H_2(BT)$, and hence the vector space $L(T)$ with $H_2(BT, \mathbb{R})$. In this way we regard vectors v_i in a multi-fan as lying in $H_2(BT, \mathbb{R})$. On the other hand a characteristic submanifold M_i with a fixed orientation determines a cohomology class ξ_i in $H^2_T(M)$, the equivariant Poincaré dual of M_i. These cohomology classes are fundamental for describing the first Chern class of an equivariant line bundle over M. This fact enables us to associate a multi-polytope and a generalization of Duistermaat-Heckman measure with an equivariant line bundle. To a T-line bundle L with the equivariant first Chern class of the form $c^T_1(L) = \sum c_i\xi_i$, we associate an arrangement of affine hyperplanes F_i in $H^2(BT; \mathbb{R}) = L(T)^*$ defined by

$$F_i = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_i\rangle = c_i\}.$$

This arrangement defines the multi-polytope associated with the line bundle L. Moreover it is possible to define the equivariant cohomology of a complete simplicial multi-fan and extend the results to such abstract multi-fans and multi-polytopes.

If $v_{i_1}, \dots, v_{i_n}$ are primitive vectors generating an n-dimensional cone in the multi-fan associated with a torus manifold, then they form a basis of $\mathrm{Hom}(S^1, T)$. However, in the definition of abstract multi-fans, this condition is not postulated. From this point of view, it is natural to deal with torus orbifolds besides torus manifolds. This can be achieved without much change technically. More importantly every complete simplicial multi-fan (precise definition will be given later) can be realized as a multi-fan associated with a torus orbifold in dimensions greater than 2. In dimensions 1 and 2, realizable multi-fans are characterized.

We now explain contents of each section. In Section 2 we give a definition of a multi-fan and introduce certain related notions. The completeness of multi-fans is most important. It is a generalization of the notion of completeness of fans. But the definition takes somewhat sophisticated form. Section 3 is devoted to the T_y-genus of a complete multi-fan. It is defined in such a way that, when the multi-fan is associated with a unitary torus manifold M, it coincides with the T_y-genus of M. In Lemma 3.1 we exhibit an equality which is an analogue of the relation between h-vectors and f-vectors in combinatorics (see e.g. [28]), and which, we hope, sheds a more insight on that relation.

In Section 4 and 5 the notion of a multi-polytope and the associated Duistermaat-Heckman function are defined. As explained above, a multi-polytope is a pair $\mathcal{P} = (\Delta, \mathcal{F})$ of an n-dimensional complete multi-fan Δ and an arrangement of hyperplanes $\mathcal{F} = \{F_i\}$

in $H^2(BT;\mathbb{R})$ with the same index set as the set of 1-dimensional cones in Δ. It is called simple if the multi-fan Δ is simplicial. The Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$ associated with a simple multi-polytope $\mathcal{P}$ is a locally constant integer-valued function with bounded support defined on the complement of the hyperplanes $\{F_i\}$. The wall crossing formula (Lemma 5.3) which describes the difference of the values of the function on adjacent components is important for later use. In Section 6 another locally constant function on the complement of the hyperplanes $\{F_i\}$ in a multi-polytope $\mathcal{P}$, called the winding number, is introduced. It satisfies a wall crossing formula entirely similar to the Duistermaat-Heckman function. When the multi-fan Δ is associated with a torus manifold or a torus orbifold M and if there is an equivariant complex line bundle L over M, then there is a simple multi-polytope $\mathcal{P}$ naturally associated with L, and the winding number $\mathrm{WN}_{\mathcal{P}}$ is closely related to the moment map of L. In fact it can be regarded as the density function of the Duistermaat-Heckman measure associated with the moment map. Theorem 6.6, main theorem in Section 6, states that the Duistermaat-Heckman function and the winding number coincide for any simple multi-polytope.

Section 7 is devoted to a generalization of the Ehrhart polynomial to multi-polytopes. If P is a convex lattice polytope and if νP denotes the multiplied polytope by a positive integer ν, then the number of lattice points $\sharp(\nu P)$ contained in νP is developed as a polynomial of ν . It is called the Ehrhart polynomial of P. The generalization to multi-polytopes is straightforward and properties similar to that of the ordinary Ehrhart polynomial hold (Theorem 7.2). If $\mathcal{P}$ is a simple lattice multi-polytope, then the associated Ehrhart polynomial $\sharp(\nu\mathcal{P})$ is defined by

$$\sharp(\nu\mathcal{P}) = \sum_{u\in H^2(BT;\mathbb{Z})} \mathrm{DH}_{\nu\mathcal{P}_+}(u),$$

where $\mathcal{P}_+$ denotes a multi-polytope obtained from $\mathcal{P}$ by a small enlargement. Lemma 7.3 is crucial for the proof of Theorem 7.2 and for the later development of the theory. Its corollary, Corollary 7.4, gives a localization formula for the Laurent polynomial $\sum_{u\in H^2(BT;\mathbb{Z})}\mathrm{DH}_{\mathcal{P}_+}(u)t^u$ regarded as a character of T. It can be considered as a combinatorial generalization of Theorem 11.1. It reduces to $\sharp\mathcal{P}$ when evaluated at the identity. Using this fact, in Section 8, a cohomological formula expressing $\sharp\mathcal{P}$ in terms of the "Todd class" of the multi-fan and the first "Chern class" of the multi-polytope is given in Theorem 8.5. The formula can be thought of as a generalization of the formula expressing the number of lattice points in a convex lattice polytope by the Riemann-Roch number of the corresponding ample line bundle. The argument is completely combinatorial. We define the equivariant cohomology $H_T^*(\Delta)$ of a multi-fan Δ which is a module over $H^*(BT)$, the index map (Gysin homomorphism) $\pi_! : H_T^*(\Delta) \to H^{*-2n}(BT)$, the cohomology $H^*(\Delta)$ of Δ and finally the evaluation on the "fundamental class". As a corollary a generalization of Khovanskii-Pukhlikov formula ([21]) for simple lattice multi-polytopes is given in Theorem 8.7.

In Section 9 it is shown how to associate a multi-fan with a torus manifold. It is also shown that the associated multi-fan is complete. Then, in Section 10, the T_y-genus of a general torus manifold is defined and is proved to coincide with the T_y-genus of the associated multi-fan in Theorem 10.1. As a corollary a formula for the signature of a torus manifold is given. In the same spirit the definition of the equivariant index of a line bundle over a general torus manifold is given in Section 11 using a localization formula which holds in the case of unitary torus manifolds. The main theorem of this section, Theorem

11.1, gives a formula describing that equivariant index using the winding number. It generalizes the results of [18], [23] and [10] as indicated before. Results of Section 5 and 6 are crucially used here.

In Section 12 necessary changes to deal with torus orbifolds are explained briefly. One of remarkable points is that the torus action and the orbifold structure is closely related for a torus orbifold as is explained in Lemma 12.3. In the last section realization problem is dealt with. Main results of the section are Theorems 13.1, 13.2 and 13.3.

2. Multi-fans

In [23], we introduced the notion of a unitary toric manifold, which contains a compact non-singular toric variety as an example, and associated with it a combinatorial object called a multi-fan, which is a more general notion than the complete non-singular fan. In this section, we define a multi-fan in a combinatorial way and in full generality. The reader will find that our notion of multi-fan is a complete generalization of a fan. We also define the completeness and non-singularity of a multi-fan, which generalize the corresponding notion of a fan. To do this, we begin with reviewing the definition of a fan.

Let N be a lattice of rank n, which is isomorphic to $\mathbb{Z}^n$. We denote the real vector space $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with apex at the origin) if there exits a finite number of vectors $v_1, \dots, v_m$ in N such that

$$\sigma = \{r_1 v_1 + \cdots + r_m v_m \mid r_i \in \mathbb{R} \text{ and } r_i \geq 0 \text{ for all } i\}$$

and $\sigma \cap (-\sigma) = \{0\}$. Here "rational" means that it is generated by vectors in the lattice N, and "strong" convexity that it contains no line through the origin. We will often call a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ simply a *cone* in N. The dimension $\dim \sigma$ of a cone σ is the dimension of the linear space spanned by vectors in σ. A subset τ of σ is called a *face* of σ if there is a linear function $\ell\colon N_{\mathbb{R}} \to \mathbb{R}$ such that ℓ takes nonnegative values on σ and $\tau = \ell^{-1}(0) \cap \sigma$. A cone is regarded as a face of itself, while others are called *proper* faces.

Definition. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Δ is also a cone in Δ;
(2) The intersection of two cones in Δ is a face of each.

Definition. A fan Δ is said to be *complete* if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$.

A cone is called *simplicial*, or a *simplex*, if it is generated by linearly independent vectors. If the generating vectors can be taken as a part of a basis of N, then the cone is called *non-singular*.

Definition. A fan Δ is said to be *simplicial* (resp. *non-singular*) if every cone in Δ is simplicial (resp. non-singular).

The basic theory of toric varieties tells us that a fan is complete (resp. simplicial or non-singular) if and only if the corresponding toric variety is compact (resp. an orbifold or non-singular).

For each $\tau \in \Delta$, we define N^τ to be the quotient lattice of N by the sublattice generated (as a group) by $\tau \cap N$; so the rank of N^τ is $n - \dim \tau$. We consider cones in Δ that contain τ as a face, and project them on $(N^\tau)_{\mathbb{R}}$. These projected cones form a fan in N^τ, which we denote by Δ_τ and call the *projected fan* with respect to τ. The dimensions of the projected cones decrease by $\dim \tau$. The completeness, simpliciality and non-singularity of Δ are inherited to Δ_τ for any τ.

We now generalize these notions of a fan. Let N be as before. Denote by $\mathrm{Cone}(N)$ the set of all cones in N. An ordinary fan is a subset of $\mathrm{Cone}(N)$. The set $\mathrm{Cone}(N)$ has a (strict) partial ordering $\prec$ defined by: $\tau \prec \sigma$ if and only if τ is a proper face of σ. The cone $\{0\}$ consisting of the origin is the unique minimum element in $\mathrm{Cone}(N)$. On the other hand, let Σ be a partially ordered finite set with a unique minimum element. We denote by the (strict) partial ordering by $<$ and the minimum element by $*$. An example of Σ used later is an abstract simplicial set with an empty set added as a member, which we call an *augmented simplicial set*. In this case the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1)-simplex. Suppose that there is a map

$$C\colon \Sigma \to \mathrm{Cone}(N)$$

such that

(1) $C(*) = \{0\}$;
(2) If $I < J$ for $I, J \in \Sigma$, then $C(I) \prec C(J)$;
(3) For any $I, J \in \Sigma$ and $\kappa \in \mathrm{Cone}(N)$ such that $I < J$ and $C(I) \prec \kappa \prec C(J)$, there is a unique element $K \in \Sigma$ such that $I < K < J$ and $C(K) = \kappa$.

For an integer m such that $0 \le m \le n$, we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \dim C(I) = m\}.$$

One can easily check that $\Sigma^{(m)}$ does not depend on C. When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m, namely I is an $(m-1)$-simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notation $|I|$ for m when $I \in \Sigma^{(m)}$.

The image $C(\Sigma)$ is a finite set of cones in N. We may think of a pair (Σ, C) as a set of cones in N labeled by the ordered set Σ. Cones in an ordinary fan intersect only at their faces, but cones in $C(\Sigma)$ may overlap, even the same cone may appear repeatedly with different labels. The pair (Σ, C) is almost what we call a multi-fan, but we incorporate a pair of weight functions on cones in $C(\Sigma)$ of the highest dimension $n = \mathrm{rank}\, N$. More precisely, we consider two functions

$$w^{\pm}\colon \Sigma^{(n)} \to \mathbb{Z}_{\ge 0}.$$

We assume that $w^+(I) > 0$ or $w^-(I) > 0$ for every $I \in \Sigma^{(n)}$. These two functions naturally arise from geometry, and their sum corresponds to Euler number while their difference is related to Todd genus (see [23]).

Definition. We call a triple $\Delta := (\Sigma, C, w^{\pm})$ a *multi-fan* in N. We define the dimension of Δ to be the rank of N (or the dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of $\mathrm{Cone}(N)$, one can view it as a multi-fan by taking $\Sigma = \Delta$, $C =$ the inclusion map, $w^+ = 1$, and $w^- = 0$. In a similar way as in the case of ordinary fans, we say that a multi-fan $\Delta = (\Sigma, C, w^{\pm})$ is *simplicial* (resp.

non-singular) if every cone in $C(\Sigma)$ is simplicial (resp. non-singular). The following lemma is easy.

Lemma 2.1. *A multi-fan $\Delta = (\Sigma, C, w^{\pm})$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

The definition of completeness of a multi-fan Δ is rather complicated. A naive definition of the completeness would be that the union of cones in $C(\Sigma)$ covers the entire space $N_{\mathbb{R}}$. But it turns out that this is not a right definition if we look at the multi-fan associated with a unitary toric manifold, see Section 9. Although the two weighted functions $w^{\pm}$ are incorporated in the definition of a multi-fan, only the difference

$$w := w^{+} - w^{-}$$

matters in this paper except Section 13. We shall introduce the following intermediate notion of pre-completeness at first. A vector $v \in N_{\mathbb{R}}$ will be called generic if v does not lie on a linear subspace spanned by a cone in $C(\Sigma)$ of dimesnsion less than n. For a generic vector v we set $d_v = \sum_{v \in C(I)} w(I)$, where the sum is understood to be zero if there is no such I.

Definition. We call a multi-fan $\Delta = (\Sigma, C, w^{\pm})$ of dimension n *pre-complete* if $\Sigma^{(n)} \neq \emptyset$ and the integer d_v is independent of the choice of generic vectors v. We call this integer the *degree* of Δ and denote it by $\deg(\Delta)$.

Remark. For an ordinary fan, pre-completeness is same as completeness.

To define the completeness for a multi-fan Δ, we need to define a projected multi-fan with respect to an element in Σ. We do it as follows. For each $K \in \Sigma$, we set

$$\Sigma_K := \{J \in \Sigma \mid K \leq J\}.$$

It inherits the partial ordering from Σ, and K is the unique minimum element in Σ_K. A map

$$C_K \colon \Sigma_K \to \mathrm{Cone}(N^{C(K)})$$

sending $J \in \Sigma_K$ to the cone $C(J)$ projected on $(N^{C(K)})_{\mathbb{R}}$ satisfies the three properties above required for C. We define two functions

$$w_K{}^{\pm} \colon \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

to be the restrictions of $w^{\pm}$ to $\Sigma_K^{(n-|K|)}$. A triple $\Delta_K := (\Sigma_K, C_K, w_K{}^{\pm})$ is a multi-fan in $N^{C(K)}$, and this is the desired *projected multi-fan* with respect to $K \in \Sigma$. When Δ is an ordinary fan, this definition agrees with the previous one.

Definition. A pre-complete multi-fan $\Delta = (\Sigma, C, w^{\pm})$ is said to be *complete* if the projected multi-fan Δ_K is pre-complete for any $K \in \Sigma$.

Remark. A multi-fan Δ is complete if and only if the projected multi-fan Δ_J is pre-complete for any $J \in \Sigma^{(n-1)}$. The argument is as follows. The pre-completeness of Δ_J for $J \in \Sigma^{(n-1)}$ implies that $d_v = \sum_{v \in C(I)} w(I)$ remains unchanged when v gets across the codimension one cone $C(J)$, which means the pre-completeness of Δ. Since $\Sigma_K^{(n-|K|-1)}$ is contained in $\Sigma^{(n-1)}$ for any $K \in \Sigma$, the pre-completeness of Δ_J for any $J \in \Sigma^{(n-1)}$ also implies the pre-completeness of Δ_K for any $K \in \Sigma$.

Example 2.2. Here is an example of a complete non-singular multi-fan of degree two. Let $v_1, \dots, v_5$ be integral vectors shown in Figure 1, where the dots denote lattice points.

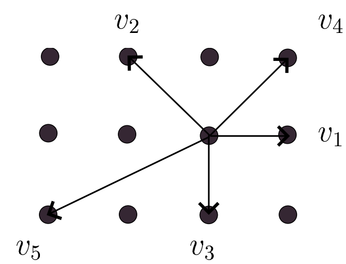

Figure 1

The vectors are rotating around the origin twice in counterclockwise. We take

$$\Sigma = \{\phi, \{1\}, \dots, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\},$$

define $C\colon \Sigma \to \mathrm{Cone}(N)$ by

$$\begin{aligned} C(\{i\}) &= \text{ the cone spanned by } v_i, \\ C(\{i, i+1\}) &= \text{ the cone spanned by } v_i \text{ and } v_{i+1}, \end{aligned}$$

where $i = 1, \dots, 5$ and 6 is understood to be 1, and take $w^{\pm}$ such that $w = 1$ on every two dimensional cone. Then $\Delta = (\Sigma, C, w^{\pm})$ is a complete non-singular two-dimensional multi-fan with $\deg(\Delta) = 2$.

Example 2.3. Here is an example of a complete multi-fan "with folds". Let $v_1, \dots, v_5$ be vectors shown in Figure 2.

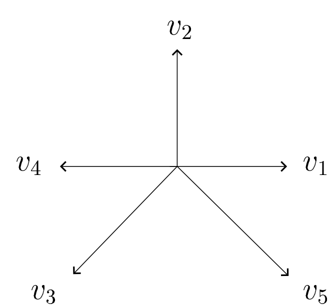

Figure 2

We take the same Σ and C as in Example 2.2 and take $w^{\pm}$ such that

$$w(\{3,4\}) = -1 \quad \text{and} \quad w(\{i, i+1\}) = 1 \ \text{ for } i \neq 3.$$

Then $\Delta = (\Sigma, C, w^{\pm})$ is a complete two-dimensional multi-fan with $\deg(\Delta) = 1$.

A similar example can be constructed for a number of vectors $v_1, \dots, v_d$ ($d \geq 3$) by defining

$$w(\{i, i+1\}) = 1 \quad \text{if } v_i \text{ and } v_{i+1} \text{ are rotating in counterclockwise,}$$
$$w(\{i, i+1\}) = -1 \quad \text{if } v_i \text{ and } v_{i+1} \text{ are rotating in clockwise,}$$

where $d+1$ is understood to be 1. The degree $\deg(\Delta)$ is the rotation number of the vectors $v_1, \dots, v_d$ around the origin in counterclockwise and may not be one.

Example 2.4. Here is an example of a multi-fan which is pre-complete but not complete. Let $v_1, \dots, v_5$ be vectors shown in Figure 3.

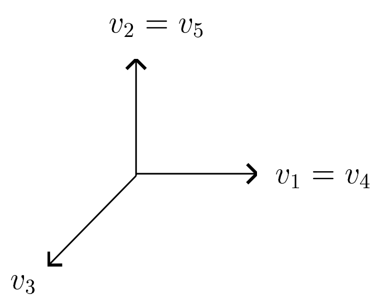

FIGURE 3

We take

$$\Sigma = \{\phi, \{1\}, \dots, \{5\}, \{1,2\}, \{2,3\}, \{3,1\}, \{4,5\}\},$$

define $C \colon \Sigma \to \mathrm{Cone}(N)$ as in Example 2.2, and take $w^{\pm}$ such that

$$w(\{1,2\}) = 2,\ w(\{2,3\}) = 1,\ w(\{3,1\}) = 1,\ w(\{4,5\}) = -1.$$

Then $\Delta = (\Sigma, C, w^{\pm})$ is a two-dimensional multi-fan which is pre-complete (in fact, $\deg(\Delta) = 1$) but not complete because the projected multi-fan $\Delta_{\{i\}}$ for $i \neq 3$ is not pre-complete.

So far, we treated *rational* cones that are generated by vectors in the lattice N. But, most of the notions introduced above make sense even if we allow cones generated by vectors in $N_{\mathbb{R}}$ which may not be in N. In fact, the notion of non-singularity requires the lattice N, but others do not. Therefore, one can define a multi-fan and its completeness and simpliciality in this extended category as well. The reader will find that the arguments developed in Sections 3 through 6 work in this extended category.

3. T_y-GENUS OF A MULTI-FAN

A unitary toric manifold M determines a complete non-singular multi-fan. (This will be discussed and extended to torus manifolds in Section 9.) On the other hand, the T_y-genus for unitary manifolds introduced by Hirzebruch in his famous book [17] is defined for M. It is a polynomial in one variable y of degree (at most) $\frac{1}{2}\dim M$. The Kosniowski formula about the T_y-genus for unitary S^1-manifolds (see [15], [19]) and the results in [23] imply that the T_y-genus of M should be described in terms of the multi-fan associated with M. In this section (and in Section 10) we give the explicit description. In fact, our argument is rather more general. We think of the T_y-genus of M as a polynomial invariant of the associated multi-fan which is complete and non-singular. It turns out that the polynomial invariant can be defined not only for the multi-fans associated with unitary toric manifolds but also for all complete simplicial multi-fans.

Since the lattice N is unnecessary from now until the end of Section 6, we shall denote the vector space, in which cones sit, by V instead of $N_{\mathbb{R}}$. Let $\Delta = (\Sigma, C, w^{\pm})$ be a complete simplicial multi-fan defined on V. By Lemma 2.1 we may assume that Σ is an augmented simplicial set, say, consisting of subsets of $\{1,\dots,d\}$ and $\Sigma^{(1)} = \{\{1\},\dots,\{d\}\}$ where d is the number of elements in $\Sigma^{(1)}$. For each $i = 1,\dots,d$, let v_i denote a nonzero vector in the one-dimensional cone $C(\{i\})$. Choose a generic element $v \in V$. Let $I \in \Sigma^{(n)}$. Since v_i's ($i \in I$) are linearly independent, v has a unique expression $\sum_{i\in I} a_i v_i$ with real numbers a_i's. The coefficients a_i's are all nonzero because v is generic. We set

$$\mu(I) := \sharp\{i \in I \mid a_i > 0\}.$$

This depends on v although v is not recorded in the notation $\mu(I)$.

Definition. For an integer q with $0 \le q \le n$, we define

$$h_q(\Delta) := \sum_{\mu(I)=q} w(I) \quad \text{and} \quad e_q(\Delta) := \sum_{K\in\Sigma^{(q)}} \deg(\Delta_K).$$

Note that $h_n(\Delta) = \deg(\Delta) = e_0(\Delta)$, and $e_q(\Delta)$'s are independent of v. If Δ is a complete simplicial multi-fan such that $\deg(\Delta) = 1$ and $w(I) = 1$ for all $I \in \Sigma^{(n)}$ (e.g. this is the case if Δ is a complete simplicial ordinary fan), then $\deg(\Delta_K)$ equals 1 for all $K \in \Sigma$ and hence $e_q(\Delta)$ agrees with the number of cones of dimension q in the multi-fan.

The following lemma reminds us of the relation between the h-vectors and the f-vectors for simplicial sets studied in combinatorics (see [28]).

Lemma 3.1. $\sum_{q=0}^{n} h_q(\Delta)(s+1)^q = \sum_{m=0}^{n} e_{n-m}(\Delta)s^m$ *where s is an indeterminate.*

Proof. The lemma is equivalent to the following identity:

$$\sum_{q=m}^{n} h_q(\Delta)\binom{q}{m} = e_{n-m}(\Delta). \tag{3.1}$$

It follows from the definition of $h_q(\Delta)$ that

$$\text{l.h.s. of (3.1)} = \sum_{q=m}^{n}\binom{q}{m}\sum_{\mu(I)=q} w(I). \tag{3.2}$$

On the other hand, we shall rewrite $e_{n-m}(\Delta)$. It follows from the definition of $\deg(\Delta_K)$ that

$$\deg(\Delta_K) = \sum_{J\in\Sigma_K^{(n-|K|)} \text{ s.t. } v_K\in C_K(J)} w_K(J)$$

where v_K denotes the projection image of v on the quotient vector space of V by the subspace V_K spanned by the cone $C(K)$. Note that v_K lies in $C_K(J)$ if and only if v lies in $C(J\cup K)$ modulo V_K, and that $w_K(J)=w(J\cup K)$ by definition. Therefore, writing $J\cup K$ as I, the identity above turns into

$$\deg(\Delta_K)=\sum_I w(I),$$

where I runs over elements in $\Sigma^{(n)}$ such that $K\subset I$ and $v\in C(I)$ modulo V_K. Putting this in the defining equation of $e_{n-m}(\Delta)$, we have

$$e_{n-m}(\Delta)=\sum_{K,I} w(I), \tag{3.3}$$

where the sum is taken over elements $K\in\Sigma^{(n-m)}$ and $I\in\Sigma^{(n)}$ such that $K\subset I$ and $v\in C(I)$ modulo V_K. Fix $I\in\Sigma^{(n)}$ with $\mu(I)=q$, and observe how many times I appears in the above sum. It is equal to the number of $K\in\Sigma^{(n-m)}$ such that $K\subset I$ and $v\in C(I)$ modulo V_K. But the number of such K is $\binom{q}{m}$. To see this, we note that $\mu(I)=q$ means that $\sharp\{i\in I\mid a_i>0\}=q$ by definition, where $v=\sum_{i\in I}a_iv_i$, and that the condition that $v\in C(I)$ modulo V_K is equivalent to saying that K contains the complement of the set $\{i\in I\mid a_i>0\}$ in I. Therefore, any such K is obtained as the complement of a subset of $\{i\in I\mid a_i>0\}$ with cardinality m, so that the number of such K is $\binom{q}{m}$. This together with (3.2) and (3.3) proves the identity (3.1). □

Corollary 3.2. (1) *$h_q(\Delta)$'s are independent of the choice of the generic vector v.*
(2) *$h_q(\Delta)=h_{n-q}(\Delta)$ for any q.*

Proof. (1) This immediately follows from Lemma 3.1 because $e_q(\Delta)$'s are independent of v.

(2) If we take $-v$ instead of v, then $\mu(I)$ turns into $n-\mu(I)$, so that $h_q(\Delta)$ turns into $h_{n-q}(\Delta)$. Since $h_q(\Delta)$'s are independent of v as shown in (1) above, this proves $h_q(\Delta)=h_{n-q}(\Delta)$. □

When Δ is associated with a unitary toric manifold M, the T_y-genus of M turns out to be given by $\sum_{q=0}^n h_q(\Delta)(-y)^q$. (This will be discussed in Section 10 later.) Motivated by this observation,

Definition. For a complete simplicial multi-fan Δ, we define

$$T_y[\Delta]:=\sum_{q=0}^n h_q(\Delta)(-y)^q$$

and call it the *T_y-genus* of Δ. Note that $T_0[\Delta]=h_0(\Delta)=h_n(\Delta)=\deg(\Delta)$.

Lemma 3.1 can be restated as

Corollary 3.3. *Let Δ be a complete simplicial multi-fan. Then*

$$T_y[\Delta]=\sum_{m=0}^n e_{n-m}(\Delta)(-1-y)^m.$$

4. Multi-polytopes

A convex polytope P in $V^* = \mathrm{Hom}(V, \mathbb{R})$ is the convex hull of a finite set of points in V^*. It is the intersection of a finite number of half spaces in V^* separated by affine hyperplanes, so there are a finite number of nonzero vectors $v_1, \dots, v_d$ in V and real numbers $c_1, \dots, c_d$ such that

$$P = \{u \in V^* \mid \langle u, v_i \rangle \leq c_i \text{ for all } i\},$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between V^* and V. (Warning: In this paper, we take v_i to be "outward normal" to the corresponding face of P contrary to the usual convention in algebraic geometry, cf. e.g. [25].) The convex polytope P can be recovered from the data $\{(v_i, c_i) \mid i = 1, \dots, d\}$. But, a more general figure like Q shaded in Figure 4 cannot be determined by the data $\{(v_i, c_i) \mid i = 1, \dots, d\}$. We need to prescribe the vertices of Q, in other words, which pairs of lines ℓ_i's are presumed to intersect. For instance, if four points $\ell_1 \cap \ell_2$, $\ell_2 \cap \ell_3$, $\ell_3 \cap \ell_4$ and $\ell_4 \cap \ell_1$ are presumed to be vertices (and the others such as $\ell_2 \cap \ell_4$ are not), then we can find the figure Q in Figure 4. But, if different four points $\ell_1 \cap \ell_4$, $\ell_4 \cap \ell_2$, $\ell_2 \cap \ell_3$ and $\ell_3 \cap \ell_1$ are presumed to be vertices, then we obtain a figure Q' shaded in Figure 4.

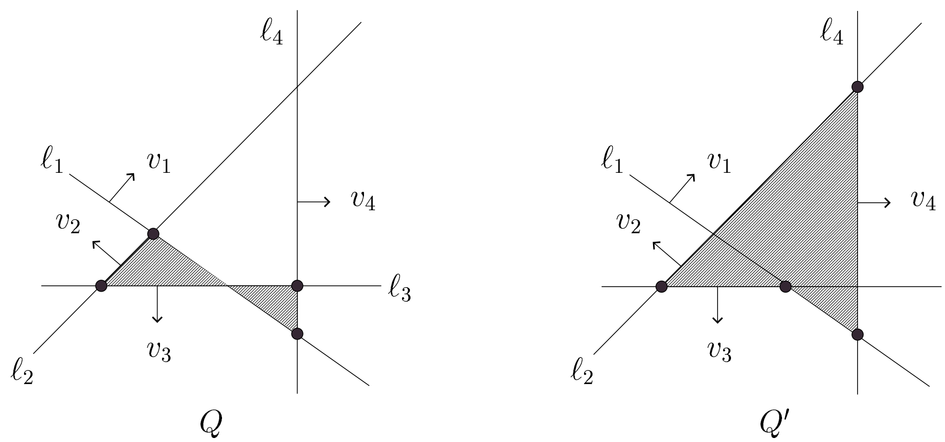

Figure 4

The data of whether two lines ℓ_i and ℓ_j are presumed to intersect is equivalent to the data of whether the corresponding vectors v_i and v_j span a cone. In the former (resp. latter) example above, resulting cones are four two-dimensional ones shown in Figure 5 (1) (resp. (2)). Needless to say, ℓ_i is 'perpendicular' to the half line spanned by v_i.

A polytope gives rise to a multi-fan in this way. One notes that a convex polytope gives rise to a complete fan. Taking this observation into account, we reverse a gear. We start with a complete multi-fan $\Delta = (\Sigma, C, w^{\pm})$. Let $\mathrm{HP}(V^*)$ be the set of all affine hyperplanes in V^*.

Definition. Let $\Delta = (\Sigma, C, w^{\pm})$ be a complete multi-fan and let $\mathcal{F} \colon \Sigma^{(1)} \to \mathrm{HP}(V^*)$ be a map such that the affine hyperplane $\mathcal{F}(I)$ is 'perpendicular' to the half line $C(I)$ for each $I \in \Sigma^{(1)}$, i.e., an element in $C(I)$ takes a constant on $\mathcal{F}(I)$. We call a pair $(\Delta, \mathcal{F})$ a *multi-polytope* and denote it by $\mathcal{P}$. The dimension of a multi-polytope $\mathcal{P}$ is defined to be the dimension of the multi-fan Δ. We say that a multi-polytope $\mathcal{P}$ is *simple* if Δ is simplicial.

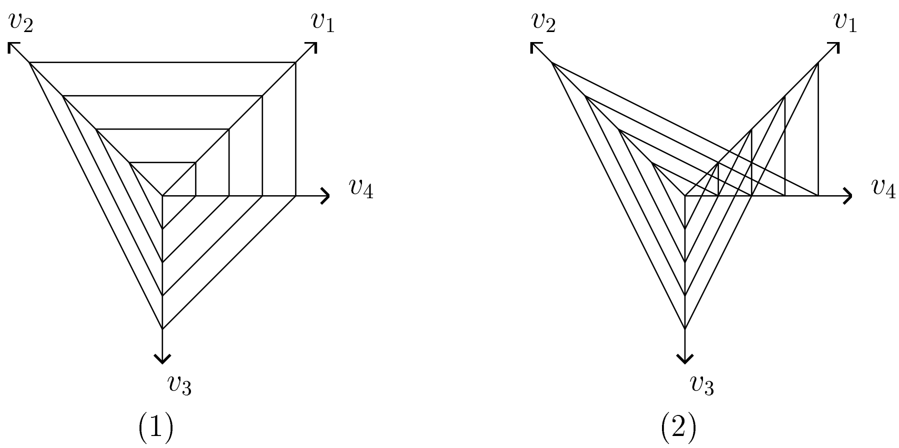

FIGURE 5

Remark. There are two notions similar to that of multi-polytopes, which were introduced by Karshon-Tolman [18] and Khovanskii-Pukhlikov [21] when Δ is an ordinary fan. They use the terminology *twisted polytope* and *virtual polytope* respectively. The notion of multi-polytopes is a direct generalization of that of twisted polytopes, and it also generalizes that of virtual polytopes, see [24].

Example 4.1. A convex polytope determines a complete fan together with an arrangement of affine hyperplanes containing the facets of the polytope (as explained above), so it uniquely determines a multi-polytope.

Example 4.2. Associated with the multi-fan in Example 2.2, one obtains the arrangement of lines drawn in Figure 6 with a suitable choice of the map $\mathcal{F}$. The pentagon shown up in Figure 6 produces the same arrangement of lines and can be viewed as a multi-polytope as explained in Example 4.1 above, but these two multi-polytopes are different because the underlying multi-fans are different; one is a multi-fan of degree two while the other is an ordinary fan. The reader will find a star-shaped figure in the former multi-polytope.

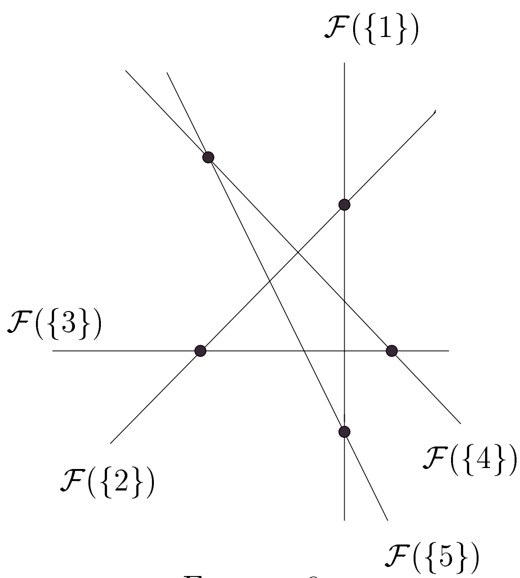

FIGURE 6

5. Duistermaat-Heckman functions

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ defines an arrangement of affine hyperplanes in V^*. In this section, we associate with $\mathcal{P}$ a function on V^* minus the affine hyperplanes when $\mathcal{P}$ is simple. This function is locally constant and Guillemin-Lerman-Sternberg formula ([11] [12]) tells us that it agrees with the density function of a Duistermaat-Heckman measure when $\mathcal{P}$ arises from a moment map.

Hereafter our multi-polytope $\mathcal{P}$ is assumed to be simple, so that the multi-fan $\Delta = (\Sigma, C, w^{\pm})$ is complete and simplicial unless otherwise stated. As before, we may assume that Σ consists of subsets of $\{1, \dots, d\}$ and $\Sigma^{(1)} = \{\{1\}, \dots, \{d\}\}$, and denote by v_i a nonzero vector in the one-dimensional cone $C(\{i\})$. To simplify notation, we denote $\mathcal{F}(\{i\})$ by F_i and set

$$F_I := \cap_{i \in I} F_i \quad \text{for } I \in \Sigma.$$

F_I is an affine space of dimension $n - |I|$. In particular, if $|I| = n$ (i.e., $I \in \Sigma^{(n)}$), then F_I is a point, denoted by u_I.

Suppose $I \in \Sigma^{(n)}$. Then the set $\{v_i \mid i \in I\}$ forms a basis of V. Denote its dual basis of V^* by $\{u_i^I \mid i \in I\}$, i.e., $\langle u_i^I, v_j \rangle = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. Take a generic vector $v \in V$ such that $\langle u_i^I, v \rangle \neq 0$ for all $I \in \Sigma^{(n)}$ and $i \in I$, and set

$$(-1)^I := (-1)^{\sharp\{i \in I \mid \langle u_i^I, v \rangle > 0\}} \quad \text{and} \quad (u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v \rangle > 0 \\ -u_i^I & \text{if } \langle u_i^I, v \rangle < 0. \end{cases}$$

We denote by $C^*(I)^+$ the cone in V^* spanned by $(u_i^I)^+$'s $(i \in I)$ with apex at u_I, and by ϕ_I its characteristic function.

Definition. We define a function $\mathrm{DH}_{\mathcal{P}}$ on $V^* \setminus \cup_{i=1}^d F_i$ by

$$\mathrm{DH}_{\mathcal{P}} := \sum_{I \in \Sigma^{(n)}} (-1)^I w(I) \phi_I$$

and call it the *Duistermaat-Heckman function* associated with $\mathcal{P}$.

Remark. Apparently, the function $\mathrm{DH}_{\mathcal{P}}$ is defined on the whole space V^* and depends on the choice of the generic vector $v \in V$, but we will see in Lemma 5.4 below that it is independent of v on $V^* \setminus \cup F_i$. This is the reason why we restricted the domain of the function to $V^* \setminus \cup F_i$.

For the moment, we shall see the independence of v when $\dim \mathcal{P} = 1$.

Example 5.1. Suppose $\dim \mathcal{P} = 1$. We identify V with $\mathbb{R}$, so that V^* is also identified with $\mathbb{R}$. Let E be the subset of $\{1, \dots, d\}$ such that $i \in E$ if and only if $C(\{i\})$ is the half line consisting of nonnegative real numbers. Then the completeness of Δ means that

$$\sum_{i \in E} w(\{i\}) = \sum_{i \notin E} w(\{i\}) = \deg(\Delta). \tag{5.1}$$

Take a nonzero vector v. Since V^* is identified with $\mathbb{R}$, each affine hyperplane F_i is nothing but a real number. Suppose that v is toward the positive direction. Then

$$(-1)^{\{i\}} = \begin{cases} -1 & \text{if } i \in E \\ 1 & \text{if } i \notin E \end{cases} \tag{5.2}$$

and the support of the characteristic function $\phi_{\{i\}}$ is the half line given by

$$\{u \in \mathbb{R} \mid F_i \le u\}.$$

Therefore

$$\mathrm{DH}_{\mathcal{P}}(u) = \sum_{i\in E \text{ s.t. } F_i<u} (-w(\{i\})) + \sum_{i\notin E \text{ s.t. } F_i<u} w(\{i\}) \tag{5.3}$$

for $u \in \mathbb{R}\setminus \cup F_i$. If u is sufficiently small, then the sum above is empty; so it is zero. If u is sufficiently large, then the sum is also zero by (5.1). Hence the support of the function $\mathrm{DH}_{\mathcal{P}}$ is bounded.

Now, suppose that v is toward the negative direction. Then $(-1)^{\{i\}}$ above is multiplied by -1 and the inequality $\le$ above turns into $\ge$. Therefore

$$\mathrm{DH}_{\mathcal{P}}(u) = \sum_{i\in E \text{ s.t. } u<F_i} w(\{i\}) + \sum_{i\notin E \text{ s.t. } u<F_i} (-w(\{i\})). \tag{5.4}$$

It follows that

$$\text{r.h.s. of (5.3)} - \text{r.h.s. of (5.4)} = -\sum_{i\in E} w(\{i\}) + \sum_{i\notin E} w(\{i\}),$$

which is zero by (5.1). This shows that the function $\mathrm{DH}_{\mathcal{P}}$ is independent of v when $\dim \mathcal{P} = 1$.

Example 5.2. For the star-shaped multi-polytope in Example 4.2, $\mathrm{DH}_{\mathcal{P}}$ takes 2 on the pentagon, 1 on the five triangles adjacent to the pentagon and 0 on other (unbounded) regions. The check is left to the reader.

For each $\{i\} \in \Sigma^{(1)}$, the projected multi-fan $\Delta_{\{i\}} = (\Sigma_{\{i\}}, C_{\{i\}}, w^{\pm}_{\{i\}})$, which we abbreviate as $\Delta_i = (\Sigma_i, C_i, w^{\pm}_i)$, is defined on the quotient vector space V/V_i of V by the one-dimensional subspace V_i spanned by v_i. Since Δ is complete and simplicial, so is Δ_i. We identify the dual space $(V/V_i)^*$ with

$$(V^*)_i := \{u \in V^* \mid \langle u, v_i\rangle = 0\}$$

in a natural way. We choose an element $f_i \in F_i$ arbitrarily and translate F_i onto $(V^*)_i$ by $-f_i$. If $\{i,j\} \in \Sigma^{(2)}$, then F_j intersects F_i and their intersection will be translated into $(V^*)_i$ by $-f_i$. This observation leads us to consider the map

$$\mathcal{F}_i \colon \Sigma_i \to \mathrm{HP}((V^*)_i)$$

sending $\{j\} \in \Sigma_i^{(1)}$ to $F_i \cap F_j$ translated by $-f_i$. The pair $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$ is a multi-polytope in $(V/V_i)^* \cong (V^*)_i$.

Let $I \in \Sigma^{(n)}$ such that $i \in I$. Since $\langle u^I_j, v_i\rangle = \delta_{ij}$, u^I_j for $j \ne i$ is an element of $(V^*)_i$, which we also regard as an element of $(V/V_i)^*$ through the isomorphism $(V/V_i)^* \cong (V^*)_i$. We denote the projection image of the generic element $v \in V$ on V/V_i by $\bar{v}$. Then we have $\langle \bar{v}, u^I_j\rangle = \langle v, u^I_j\rangle$ for $j \ne i$, where u^I_j at the left-hand side is viewed as an element of $(V/V_i)^*$ while the one at the right-hand side is viewed as an element of $(V^*)_i$. Since $\langle \bar{v}, u^I_j\rangle = \langle v, u^I_j\rangle \ne 0$ for $j \ne i$, we use $\bar{v}$ to define $\mathrm{DH}_{\mathcal{P}_i}$.

Lemma 5.3. (Wall crossing formula.) *Let F be one of F_i's. Let u_α and u_β be elements in $V^*\setminus \cup_{i=1}^d F_i$ such that the segment from u_α to u_β intersects the wall F transversely at*

μ, and does not intersect any other $F_j \neq F$. Then

$$\mathrm{DH}_{\mathcal{P}}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) = \sum_{i:F_i=F} \operatorname{sign}\langle u_\beta - u_\alpha, v_i\rangle\, \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i).$$

Proof. For simplicity we assume that there is only one i such that $F_i = F$. We may assume that $\langle u_\beta - u_\alpha, v_i\rangle$ is positive without loss of generality. The situation is as in Figure 7.

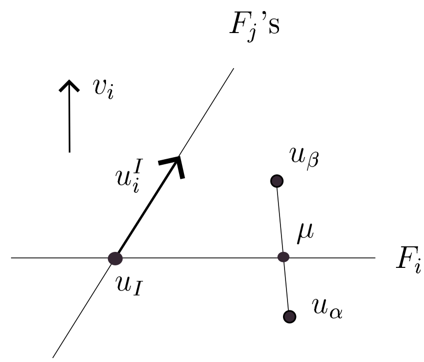

FIGURE 7

It follows from the definition of $\mathrm{DH}_{\mathcal{P}}$ that the difference between $\mathrm{DH}_{\mathcal{P}}(u_\alpha)$ and $\mathrm{DH}_{\mathcal{P}}(u_\beta)$ arises from the cones $C^*(I)^+$'s for $I \in \Sigma^{(n)}$ such that $i \in I$ and $\langle u_I, v\rangle < \langle \mu, v\rangle$. In fact, one sees that

$$\mathrm{DH}_{\mathcal{P}}(u_\alpha) + \sum_I \operatorname{sign}\langle u_i^I, v\rangle(-1)^I w(I)\phi_I(\mu) = \mathrm{DH}_{\mathcal{P}}(u_\beta)$$

where I runs over the elements as above. Since $\operatorname{sign}\langle u_i^I, v\rangle(-1)^I = -(-1)^{I\backslash\{i\}}$ and $w(I) = w_i(I\backslash\{i\})$, the identity above turns into

$$\mathrm{DH}_{\mathcal{P}}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) = \sum_I (-1)^{I\backslash\{i\}} w_i(I\backslash\{i\})\phi_I(\mu).$$

Here $\phi_I(\mu)$ may be viewed as the value at μ of the characteristic function of the cones in F_i with apex u_I spanned by $(u_j^I)^+$'s $(j \in I, j \neq i)$. This shows that the right-hand side at the identity above agrees with $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i)$, proving the lemma. □

Lemma 5.4. *The support of the function $\mathrm{DH}_{\mathcal{P}}$ is bounded, and the function is independent of the choice of the generic element $v \in V$.*

Proof. Induction on the dimension of simple multi-polytopes $\mathcal{P}$. We have observed the lemma in Example 5.1 when $\dim \mathcal{P} = 1$. Suppose that the lemma is true for simple multi-polytopes of dimension $n-1$, and suppose $\dim \mathcal{P} = n$. Then the support of $\mathrm{DH}_{\mathcal{P}_i}$ is bounded by the induction assumption. This together with Lemma 5.3 implies that $\mathrm{DH}_{\mathcal{P}}$ takes the same constant on unbounded regions in $V^*\backslash \cup F_i$. On the other hand, it follows from the definition of $\mathrm{DH}_{\mathcal{P}}$ that $\mathrm{DH}_{\mathcal{P}}$ vanishes on a half space $H_r := \{u \in V^* \mid \langle u, v\rangle < r\}$ for a sufficiently small real number r, because for each $I \in \Sigma^{(n)}$ the cone $C^*(I)^+$ is contained in the complement of H_r if r is sufficiently small. Therefore the constant which $\mathrm{DH}_{\mathcal{P}}$ takes on the unbounded regions in $V^*\backslash \cup F_i$ is zero, proving the former assertion in the lemma.

As for the latter assertion in the lemma, it follows from the induction assumption that the right-hand side of the wall crossing formula in Lemma 5.3 is independent of v, and we have seen above that $\mathrm{DH}_{\mathcal{P}}$ vanishes on unbounded regions regardless of the choice of v. Thus, it follows from Lemma 5.3 that $\mathrm{DH}_{\mathcal{P}}$ is independent of v on any regions of $V^*\backslash \cup F_i$. □

6. Winding numbers

We continue to assume that our multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ is simple and that Σ is an augmented simplicial set consisting of subsets of $\{1, \dots, d\}$. In this section, we associate another locally constant function on $V^*\backslash \cup F_i$ with $\mathcal{P}$ from a topological viewpoint, and show that it agrees with the Duistermaat-Heckman function defined in Section 5.

Choose an orientation on V and fix it. We define an orientation on $I = \{i_1, \dots, i_n\} \in \Sigma^{(n)}$ as follows. If an ordered basis $(v_{i_1}, \dots, v_{i_n})$ gives the chosen orientation on V, then we say that the oriented simplex $\langle i_1, \dots, i_n\rangle$ has a positive orientation, and otherwise a negative orientation. We define

$$\langle I\rangle := \begin{cases} \langle i_1, \dots, i_n\rangle & \text{if } \langle i_1, \dots, i_n\rangle \text{ has a positive orientation,} \\ -\langle i_1, \dots, i_n\rangle & \text{if } \langle i_1, \dots, i_n\rangle \text{ has a negative orientation.} \end{cases}$$

The completeness of Δ (equivalently, the pre-completeness of the projected multi-fan Δ_J for any $J \in \Sigma^{(n-1)}$) implies that

$$\sum_{I \in \Sigma^{(n)}} w(I)\langle I\rangle$$

is a cycle. In fact, the converse holds, i.e., the completeness of Δ is equivalent to $\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle$ being a cycle. We denote by $[\Delta]$ the homology class that the cycle defines in $H_{n-1}(\Sigma)$. Actually $[\Delta]$ lies in the reduced homology $\tilde{H}_{n-1}(\Sigma)$, see Example 6.3 discussed later.

Let S be the realization of the first barycentric subdivision of Σ. For each $i \in \{1, \dots, d\}$, we denote by S_i the union of simplicies in S which contain the vertex $\{i\}$, and by S_I the intersection $\cap_{i\in I} S_i$ for $I \in \Sigma$. Note that ∂S_i can be identified with the realization of the first barycentric subdivision of Σ_i, where Σ_i is the augmented simplicial set of the projected multi-fan $\Delta_i = (\Sigma_i, C_i, w_i^{\pm})$.

The projected multi-fan Δ_i is defined on V/V_i where V_i is the one-dimensional subspace spanned by v_i. We orient V/V_i as follows: if an ordered basis $(v_i, v_{j_1}, \dots, v_{j_{n-1}})$ defines the given orientation on V, then we give V/V_i the orientation determined by $(v_{j_1}, \dots, v_{j_{n-1}})$, and otherwise give the opposite orientaiton. Then $[\Delta_i]$ is defined in $\tilde{H}_{n-2}(\Sigma_i) = \tilde{H}_{n-2}(\partial S_i)$.

Lemma 6.1. $[\Delta]$ *maps to* $[\Delta_i]$ *through the composition of maps*

$$\tilde{H}_{n-1}(\Sigma) = \tilde{H}_{n-1}(S) \xrightarrow{\iota_*} H_{n-1}(S, S\backslash \operatorname{Int} S_i) \xleftarrow[\cong]{excision} H_{n-1}(S_i, \partial S_i) \xrightarrow[\cong]{\partial} \tilde{H}_{n-2}(\partial S_i),$$

where ι *is the inclusion.*

Proof. Through ι_* and the inverse of the excision isomorphism, the cycle $\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle$ maps to $\sum_{i\in I\in\Sigma^{(n)}} w(I)\langle I\rangle$. We express $\langle I\rangle$ as $\epsilon\langle i, j_1, \dots, j_{n-1}\rangle$ where $\epsilon = +1$ or -1 and define an oriented $(n-2)$-simplex $\langle I\backslash\{i\}\rangle$ in $\Sigma_i^{(n-1)}$ by $\epsilon\langle j_1, \dots, j_{n-1}\rangle$. It follows that

$$\partial\Big(\sum_{i\in I\in\Sigma^{(n)}} w(I)\langle I\rangle\Big) = \sum_{i\in I\in\Sigma^{(n)}} w(I)\langle I\backslash\{i\}\rangle.$$

Here $w(I) = w_i(I\backslash\{i\})$ by the definition of w_i, and $i \in I \in \Sigma^{(n)}$ if and only of $I\backslash\{i\} \in \Sigma_i^{(n-1)}$. Therefore, the right-hand side above reduces to $\sum_{J\in\Sigma_i^{(n-1)}} w_i(J)\langle J\rangle$, that is $[\Delta_i]$ in $\tilde{H}_{n-2}(\partial S_i)$. □

The following lemma will be used later several times.

Lemma 6.2. *Let X and Y be topological spaces with subspaces X_i of X and Y_i of Y for each $i \in \Sigma^{(1)}$. For $I \in \Sigma$, we set $X_I := \cap_{i\in I} X_i$ and $Y_I := \cap_{i\in I} Y_i$. If*

(1) $X = \cup_{i=1}^d X_i$,
(2) *X_I's for $I \in \Sigma^{(n)}$ are disjoint, and*
(3) *Y_I is nonempty and contractible for any non-empty set $I \in \Sigma$,*

then there is a continuous map $\psi\colon X \to Y$ sending the stratum X_I to Y_I for each $I \in \Sigma$, and such a map is unique up to homotopy preserving the stratifications.

Proof. Existence. We will construct ψ inductively using decending induction on $|I|$. If $|I| = n$, then we map X_I to any point in Y_I. Thus ψ is defined on $\cup_{|I|=n} X_I$ with the image in $\cup_{|I|=n} Y_I$. Let k be a nonnegative integer less than n and $|I| = k$. Suppose that ψ is defined on $\cup_{|J|\ge k+1} X_J$ with the image in $\cup_{|J|\ge k+1} Y_J$. Then

$$\psi\colon X_I \cap (\cup_{|J|\ge k+1} X_J) \to Y_I \cap (\cup_{|J|\ge k+1} Y_J) \subset Y_I$$

extends to a continuous map from X_I to Y_I because Y_I is contractible. Thus ψ is defined on $\cup_{|I|\ge k} X_I$ with the image in $\cup_{|I|\ge k} Y_I$. This completes the induction step, so that we obtain the desired map ψ defined on X.

Uniqueness. We construct a homotopy $H : X \times [0, 1] \to Y$ of given two maps ψ_0 and ψ_1 in the lemma. The argument is almost same as above. Since Y_I is contractible, H can be defined on $\cup_{|I|=n} X_I \times [0, 1]$ with $\cup_{|I|=n} Y_I$ as the image. Let k be as above and $|I| = k$. Suppose that H is defined on $(\cup_{|J|\ge k+1} X_J) \times [0, 1]$ with the image in $\cup_{|J|\ge k+1} Y_J$ and that H agrees with ψ_t on $(\cup_{|J|\ge k+1} X_J) \times \{t\}$ for $t = 0, 1$. Then a map

$$\begin{aligned} H \cup \psi_0 \cup \psi_1\colon & (X_I \cap (\cup_{|J|\ge k+1} X_J)) \times [0, 1] \cup X_I \times \{0\} \cup X_I \times \{1\} \\ & \to (Y_I \cap (\cup_{|J|\ge k+1} Y_J)) \cup Y_I \cup Y_I = Y_I \end{aligned}$$

extends to a continuous map from $X_I \times [0, 1]$ to Y_I because Y_I is contractible. Thus H is defined on $(\cup_{|I|\ge k} X_I) \times [0, 1]$ with the image in $\cup_{|I|\ge k} Y_I$. This completes the induction step, so that we obtain the desired homotopy H defined on $X \times [0, 1]$. □

Lemma 6.2 can be applied with $X = S$, $X_i = S_i$, $Y = V^*$ and $Y_i = F_i$. It follows that the multi-polytope $\mathcal{P}$ associates a continuous map

$$\Psi\colon S \to \cup_{i=1}^d F_i \subset V^*$$

sending S_I to F_I for each $I \in \Sigma$ by Lemma 6.2, and Ψ induces a homomorphism

$$\Psi_*\colon \tilde{H}_{n-1}(S) = \tilde{H}_{n-1}(\Sigma) \to \tilde{H}_{n-1}(V^*\backslash\{u\})$$

for each $u \in V^*\backslash \cup F_i$. Such Ψ was first introduced in [14] and plays the role of a moment map. The orientation on V chosen at the beginning of this section induces an orientation on V^* in a natural way. This determines a fundamental class in $H_n(V^*, V^*\backslash\{u\})$ and hence in $\tilde{H}_{n-1}(V^*\backslash\{u\})$ through $\partial\colon H_n(V^*, V^*\backslash\{u\}) \cong \tilde{H}_{n-1}(V^*\backslash\{u\})$. We denote the fundamental class in $\tilde{H}_{n-1}(V^*\backslash\{u\})$ by $[V^*\backslash\{u\}]$.

Definition. For each $u \in V^*\backslash \cup F_i$, we define an integer $\mathrm{WN}_{\mathcal{P}}(u)$ by

$$\Psi_*([\Delta]) = \mathrm{WN}_{\mathcal{P}}(u)[V^*\backslash\{u\}]$$

and call it the *winding number* of the multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ around u.

Remark. The function $\mathrm{WN}_{\mathcal{P}}$ is independent of the choice of an orientation on V because if the orientation on V is reversed, then $[\Delta]$ and $[V^*\backslash\{u\}]$ are multiplied by -1 simultaneously. Moreover, it is locally constant and vanishes on unbounded regions separated by F_i's, which immediately follows from the definition of $\mathrm{WN}_{\mathcal{P}}$.

We will see in Theorem 6.6 below that $\mathrm{WN}_{\mathcal{P}} = \mathrm{DH}_{\mathcal{P}}$. For the moment, we shall check this coincidence when $\dim \mathcal{P} = 1$.

Example 6.3. We use the notation in Example 5.1. We identify V with $\mathbb{R}$, so that V^* is also identified with $\mathbb{R}$. Then V and V^* have standard orientations, and since v_i gives the orientation on V if and only if $i \in E$, the cycle which defines $[\Delta]$ is given by

$$\sum_{i\in E} w(\{i\})\langle i\rangle + \sum_{i\notin E} w(\{i\})(-\langle i\rangle) = -\sum_{i=1}^{d}(-1)^{\{i\}}w(\{i\})\langle i\rangle$$

where $(-1)^{\{i\}}$ is the same as in (5.2). Since Δ is complete, $\sum_{i=1}^{d}(-1)^{\{i\}}w(\{i\}) = 0$; so $[\Delta]$ actually lies in $\tilde{H}_0(\Sigma) = \tilde{H}_0(S)$ and one can rewrite the cycle above as

$$\sum_{i=1}^{d}(-1)^{\{i\}}w(\{i\})(\langle j\rangle - \langle i\rangle)$$

for any $j \in \{1, \dots, d\}$. Since $S_i = \{i\}$ and $\Psi(\{i\}) = F_i$, $\mathrm{WN}_{\mathcal{P}}(u) = 0$ unless u is in between the minimum value and the maximum value of $\{F_1, \dots, F_d\}$. Suppose u is in between them and take j such that F_j is the maximum. Then one easily sees that

$$\mathrm{WN}_{\mathcal{P}}(u) = \sum_{F_i<u}(-1)^{\{i\}}w(\{i\}).$$

This together with (5.3) shows that $\mathrm{WN}_{\mathcal{P}} = \mathrm{DH}_{\mathcal{P}}$ when $\dim \mathcal{P} = 1$. □

We will show that WN satisfies the same wall crossing formula as in Lemma 5.3. For that, we first state a lemma which expresses the winding number as a sum of local winding numbers so to speak. We orient F_i in such a way that the juxtaposition of a normal vector to F_i, whose evaluation on v_i is positive, and the orientation on F_i agrees with the prescribed orientation on V^*. By Lemma 6.2, Ψ maps a pair $(S_i, \partial S_i)$ into a pair $(F_i, F_i\backslash\{\mu\})$ for any $\mu \in F_i\backslash(F_i \cap (\cup_{j\in\Sigma_i^{(1)}} F_j)$. If we identify F_i with $(V^*)_i$ through the translation by $-f_i$ as before, then the map Ψ restricted to ∂S_i agrees with the map (up to homotopy) constructed from the multi-polytope $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$. It follows that

$$\Psi_*([\Delta_i]) = \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i)[F_i\backslash\{\mu\}]. \tag{6.1}$$

Let $u \in V^*\backslash \cup F_i$. We choose a generic ray R starting from u with direction $\gamma \in V^*$, so that the intersection $F_i \cap R$ is one point for each i if it is nonempty. We denote the point $F_i \cap R$ by R_i.

Lemma 6.4. $\mathrm{WN}_{\mathcal{P}}(u) = \sum_{i:F_i\cap R\neq\phi} \mathrm{sign}\langle\gamma, v_i\rangle\, \mathrm{WN}_{\mathcal{P}_i}(R_i - f_i).$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde H_{n-1}(S) & \to & H_{n-1}(S, S\setminus \cup_i \operatorname{Int} S_i) & \xleftarrow[\cong]{\text{excision}} & \bigoplus_i H_{n-1}(S_i, \partial S_i) & \xrightarrow[\cong]{\partial} & \bigoplus_i \tilde H_{n-2}(\partial S_i) \\
\Psi_*\downarrow & & \Psi_*\downarrow & & \Psi_*\downarrow & & \Psi_*\downarrow \\
\tilde H_{n-1}(V^*\setminus\{u\}) & \xrightarrow[\cong]{} & H_{n-1}(V^*\setminus\{u\}, V^*\setminus R) & \leftarrow & \bigoplus_i H_{n-1}(F_i, F_i\setminus\{R_i\}) & \xrightarrow[\cong]{\partial} & \bigoplus_i \tilde H_{n-2}(F_i\setminus\{R_i\})
\end{array}$$

where i runs over the indices of F_i's which intersect R. The element $[\Delta] \in \tilde H_{n-1}(S)$ maps to $\oplus_i[\Delta_i] \in \oplus_i \tilde H_{n-2}(\partial S_i)$ through the upper horizontal sequence by Lemma 6.1 and down to $\oplus_i \mathrm{WN}_{\mathcal{P}_i}(R_i - f_i)[F_i\setminus\{R_i\}]$ by (6.1).

Now we trace the lower horizontal sequence from the right to the left. Through the inverse of ∂, $[F_i\setminus\{R_i\}]$ maps to the fundamental class $[F_i, F_i\setminus\{R_i\}]$, and further maps to $\mathrm{sign}\langle\gamma, v_i\rangle[V^*\setminus\{u\}] \in \tilde H_{n-1}(V^*\setminus\{u\})$, where the sign arises from the choice of the orientation on F_i. These together with the commutativity of the diagram above show that

$$\Psi_*([\Delta]) = \sum_{i:F_i\cap R\neq\phi} \mathrm{sign}\langle\gamma, v_i\rangle\, \mathrm{WN}_{\mathcal{P}_i}(R_i - f_i)[V^*\setminus\{u\}].$$

On the other hand, $\Psi_*([\Delta]) = \mathrm{WN}_{\mathcal{P}}(u)[V^*\setminus\{u\}]$ by definition. The lemma follows by comparing these two identities. □

Lemma 6.5. *The wall crossing formula as in Lemma 5.3 holds for* WN *instead of* DH.

Proof. Subtract the identity in Lemma 6.4 for $u = u_\beta$ from that for $u = u_\alpha$. Since one can take γ to be $u_\beta - u_\alpha$, the lemma follows. □

Theorem 6.6. $\mathrm{DH}_{\mathcal{P}} = \mathrm{WN}_{\mathcal{P}}$ *for any simple multi-polytope* $\mathcal{P}$.

Proof. The identity is established in Example 6.3 when $\dim \mathcal{P} = 1$. Suppose the identity holds for simple multi-polytopes of dimension $n-1$, and suppose $\dim \mathcal{P} = n$. Both $\mathrm{DH}_{\mathcal{P}}$ and $\mathrm{WN}_{\mathcal{P}}$ are locally constant, satisfy the same wall crossing formula (Lemma 5.3, Lemma 6.5) and $\mathrm{DH}_{\mathcal{P}_i} = \mathrm{WN}_{\mathcal{P}_i}$ by induction assumption. Therefore, it suffices to see that $\mathrm{DH}_{\mathcal{P}}$ and $\mathrm{WN}_{\mathcal{P}}$ agree on one region. But we know that they vanish on unbounded regions (Lemma 5.4 and the remark after the definition of $\mathrm{WN}_{\mathcal{P}}$), hence they agree on the whole domain. This completes the induction step, proving the theorem. □

7. Ehrhart polynomials

Let P be a convex lattice polytope of dimension n in V^*, where "lattice polytope" means that each vertex of P lies in the lattice $N^* = \mathrm{Hom}(N, \mathbb{Z})$ of $V^* = \mathrm{Hom}(V, \mathbb{R})$. For a positive integer ν, let $\nu P := \{\nu u \mid u \in P\}$. It is again a convex lattice polytope in V^*. We denote by $\sharp(\nu P)$ (resp. $\sharp(\nu P^\circ)$) the number of lattice points in νP (resp. in the interior of νP). The lattice N^* determines a volume element on V^* by requiring that the volume of the unit cube determined by a basis of N^* is 1. Thus the volume of P, denoted by $\mathrm{vol}(P)$, is defined. The following theorem is well known.

Theorem 7.1. (See [9], [25] for example.) *Let P be an n-dimensional convex lattice polytope.*

(1) *$\sharp(\nu P)$ and $\sharp(\nu P^\circ)$ are polynomials in ν of degree n.*
(2) *$\sharp(\nu P^\circ) = (-1)^n \sharp(-\nu P)$, where $\sharp(-\nu P)$ denotes the polynomial $\sharp(\nu P)$ with ν replaced by $-\nu$.*
(3) *The coefficient of ν^n in $\sharp(\nu P)$ is $\mathrm{vol}(P)$ and the constant term in $\sharp(\nu P)$ is 1.*

The fan Δ associated with P may not be simplicial, but if we subdivide Δ, then we can always take a simplicial fan that is compatible with P. In this section, we show that the theorem above holds for a *simple* lattice multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$. For that, we need to define $\sharp(\mathcal{P})$ and $\sharp(\mathcal{P}^\circ)$. This is done as follows. Let v_i $(i = 1, \dots, d)$ be a primitive integral vector in the half line $C(\{i\})$. In our convention, v_i is chosen "outward normal" to the face $\mathcal{F}(\{i\})$ when $\mathcal{P}$ arises from a convex polytope. We slightly move $\mathcal{F}(\{i\})$ in the direction of v_i (resp. $-v_i$) for each i, so that we obtain a map $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) : $\Sigma^{(1)} \to \mathrm{HP}(V^*)$. We denote the multi-polytopes $(\Delta, \mathcal{F}_+)$ and $(\Delta, \mathcal{F}_-)$ by $\mathcal{P}_+$ and $\mathcal{P}_-$ respectively. Since the affine hyperplanes $\mathcal{F}_\pm(\{i\})$'s miss the lattice N^*, the functions $\mathrm{DH}_{\mathcal{P}_\pm}$ and $\mathrm{WN}_{\mathcal{P}_\pm}$ are defined on N^*.

Definition. We define

$$\sharp(\mathcal{P}) := \sum_{u \in N^*} \mathrm{DH}_{\mathcal{P}_+}(u) = \sum_{u \in N^*} \mathrm{WN}_{\mathcal{P}_+}(u),$$
$$\sharp(\mathcal{P}^\circ) := \sum_{u \in N^*} \mathrm{DH}_{\mathcal{P}_-}(u) = \sum_{u \in N^*} \mathrm{WN}_{\mathcal{P}_-}(u).$$

When $\mathcal{P}$ arises from a convex polytope P, $\mathrm{DH}_{\mathcal{P}_+} = \mathrm{WN}_{\mathcal{P}_+}$ (resp. $\mathrm{DH}_{\mathcal{P}_-} = \mathrm{WN}_{\mathcal{P}_-}$) takes 1 on u in P (resp. in the interior of P) and 0 otherwise. Therefore, $\sharp(\mathcal{P})$ (resp. $\sharp(\mathcal{P}^\circ)$) agrees with the number of lattice points in P (resp. in the interior of P) in this case.

Denote the volume element on V^* by dV^*, and define the volume $\mathrm{vol}(\mathcal{P})$ of $\mathcal{P}$ by

$$\mathrm{vol}(\mathcal{P}) := \int_{V^*} \mathrm{DH}_{\mathcal{P}}\, dV^* = \int_{V^*} \mathrm{WN}_{\mathcal{P}}\, dV^*.$$

When $\mathcal{P}$ arises from a (convex) polytope P, $\mathrm{vol}(\mathcal{P})$ agrees with the actual volume of P, but otherwise it can be zero or negative.

For a (not necessarily positive) integer ν, we denote $(\Delta, \nu\mathcal{F})$ by $\nu\mathcal{P}$, where

$$(\nu\mathcal{F})(\{i\}) := \{u \in V^* \mid \langle u, v_i \rangle = \nu c_i\}$$

when $\mathcal{F}(\{i\}) = \{u \in V^* \mid \langle u, v_i \rangle = c_i\}$ for a constant c_i.

Theorem 7.2. *Let $\mathcal{P} = (\Delta, \mathcal{F})$ be a simple lattice multi-polytope of dimension n.*

(1) *$\sharp(\nu\mathcal{P})$ and $\sharp(\nu\mathcal{P}^\circ)$ are polynomials in ν of degree (at most) n.*
(2) *$\sharp(\nu\mathcal{P}^\circ) = (-1)^n \sharp(-\nu\mathcal{P})$ for any integer ν.*
(3) *The coefficient of ν^n in $\sharp(\nu\mathcal{P})$ is $\mathrm{vol}(\mathcal{P})$ and the constant term in $\sharp(\nu\mathcal{P})$ is $\deg(\Delta)$. (See Section 2 for $\deg(\Delta)$.)*

In order to prove this theorem, we need some notations and a lemma. Basic ideas in the following arguments are in [4] and [5]. Let $I \in \Sigma^{(n)}$. Although the integral vectors $\{v_i \mid i \in I\}$ are not necessarily a basis of the lattice N, they are linearly independent. Therefore, the sublattice N_I of N generated by v_i's $(i \in I)$ is of the same rank as N, hence N/N_I is a finite group. Needless to say, N/N_I is trivial for any $I \in \Sigma^{(n)}$ if Δ is non-singular. For $u \in N_I^* = \mathrm{Hom}(N_I, \mathbb{Z}) \supset N^*$ and $g \in N/N_I$, we define

$$\chi_I(u,g) := \exp(2\pi\sqrt{-1}\langle u, v_g\rangle) \tag{7.1}$$

where $v_g \in N$ is a representative of g. The right-hand side does not depend on the choice of the representative v_g, and $\chi_I(u,\ )$ (resp. $\chi(\ ,g)$) is a homomorphism from N/N_I (resp. N_I^*) to $\mathbb{C}^*$. Note that $\chi_I(u,\ )\colon N/N_I \to \mathbb{C}^*$ is trivial if and only if $u \in N^*$. It follows that

$$\sum_{g\in N/N_I} \chi_I(u,g) = \begin{cases} |N/N_I| & \text{if } u \in N^*, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

Lemma 7.3. *For each $I \in \Sigma^{(n)}$ let u_I be the corresponding vertex of $\mathcal{P}$ and let $\{u_i^I \mid i \in I\}$ be the dual basis of $\{v_i \mid i \in I\}$ as in Section 5. Then, for $v \in N$ such that $\langle u_i^I, v\rangle$ is a nonzero integer for any $I \in \Sigma^{(n)}$ and $i \in I$, we have*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)z^{\langle u_I,v\rangle}}{|N/N_I|} \sum_{g\in N/N_I} \frac{1}{\prod_{i\in I}(1-\chi_I(u_i^I,g)z^{-\langle u_i^I,v\rangle})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u) z^{\langle u,v\rangle}$$

as functions of $z \in \mathbb{C}$.

Proof. The Maclaurin expansion of $1/(1-az^{-m})$ $(a \in \mathbb{C}^*,\ m \in \mathbb{Z})$ is given by

$$\begin{cases} -a^{-1}z^m - a^{-2}z^{2m} - \dots & \text{if } m > 0 \\ 1 + az^{-m} + a^2z^{-2m} + \dots & \text{if } m < 0. \end{cases}$$

Taking this into account, we expand the sum

$$S_I := \sum_{g\in N/N_I} \frac{1}{\prod_{i\in I}(1-\chi_I(u_i^I,g)z^{-\langle u_i^I,v\rangle})}$$

into Maclaurin series and get

$$\begin{aligned} S_I &= \sum_{g\in N/N_I} (-1)^I \prod_{i\in I} \sum_{\{b_i\}} (\chi_I(u_i^I,g)^{-b_i} z^{b_i\langle u_i^I,v\rangle}) \\ &= \sum_{g\in N/N_I} (-1)^I \sum_{\{b_i\}} \chi_I(-\sum_{i\in I} b_i u_i^I, g) z^{\langle \sum_{i\in I} b_i u_i^I, v\rangle}, \end{aligned}$$

where the summation $\sum_{\{b_i\}}$ runs over the collection of such $\{b_i \mid i \in I,\ b_i \in \mathbb{Z}\}$ that

$$b_i \geq 1 \text{ for } i \text{ with } \langle u_i^I, v\rangle > 0 \text{ and } b_i \leq 0 \text{ for } i \text{ with } \langle u_i^I, v\rangle < 0, \tag{7.3}$$

(see Section 5 for $(-1)^I$). Since

$$\sum_{g\in N/N_I} \chi_I(-\sum_{i\in I} b_i u_i^I, g) = \begin{cases} |N/N_I| & \text{if } \sum_{i\in I} b_i u_i^I \in N^*, \\ 0 & \text{otherwise,} \end{cases}$$

by (7.2), the coefficient of $z^{\langle u,v\rangle}$ for $u \in N^*$ in the Maclaurin expansion of the left-hand side at the identity in our lemma is given by

$$\sum_{I\in\Sigma^{(n)}} (-1)^I w(I)\phi_I'(u)$$

where

$$\phi'_I(u) = \begin{cases} 1 & \text{if } u = u_I + \sum_{i\in I} b_i u_i^I \text{ and } b_i\text{'s are as in (7.3),} \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that $\sum_{I\in\Sigma^{(n)}}(-1)^I w(I)\phi'_I(u)$ agrees with $\mathrm{DH}_{\mathcal{P}_+}(u)$, proving the lemma. □

Proof of Theorem 7.2. We shall prove (2) first. It suffices to prove $\sharp(\mathcal{P}^\circ) = (-1)^n \sharp(-\mathcal{P})$. Since $\sharp(\mathcal{P}^\circ) = \sum_{u\in N^*} \mathrm{WN}_{\mathcal{P}_-}(u)$ by definition, it suffices to prove that

$$\mathrm{WN}_{\mathcal{P}_-}(u) = (-1)^n \,\mathrm{WN}_{(-\mathcal{P})_+}(u) \quad \text{for any } u \in N^*. \tag{7.4}$$

Let $\Psi_{\mathcal{P}_-}$ and $\Psi_{(-\mathcal{P})_+}$ be the maps introduced in Section 6 which are associated with multi-polytopes $\mathcal{P}_-$ and $(-\mathcal{P})_+$ respectively. We note that $\Psi_{\mathcal{P}_-}$ and $-\Psi_{(-\mathcal{P})_+}$ considered as maps from S to $V^*\backslash\{u\}$ for $u \in N^*$ are homotopic. Since the multiplication by -1 on V^* sends the fundamental class $[V^*\backslash\{-u\}]$ to $(-1)^n[V^*\backslash\{u\}]$, we obtain (7.4).

We shall prove (1). Because of (2), it suffices to prove (1) for $\sharp(\nu\mathcal{P})$. We apply Lemma 7.3 to $\nu\mathcal{P}$ in place of $\mathcal{P}$ (so that u_I is replaced by νu_I), and approach z to 1 at the identity. Since the right-hand side approaches $\sharp(\nu\mathcal{P})$, it suffices to show that the left-hand side approaches a polynomial in ν of degree at most n. When $g \in N/N_I$ is the identity element, $\chi_I(u_i^I, g) = 1$. Therefore, the term in the summand $\sum_{g\in N/N_I}$ at the identity in Lemma 7.3 has a pole at $z = 1$ of degree exactly n when g is the identity element, and of degree at most n otherwise. Thus the left-hand side of the identity in Lemma 7.3 applied to $\nu\mathcal{P}$ can be written as

$$\frac{\sum_{I\in\Sigma^{(n)}} z^{\nu\langle u_I, v\rangle} h_I(z)}{(1-z)^n f(z)}$$

where $h_I(z)$ and $f(z)$ are polynomials in z and $f(1) \neq 0$. Then the repeated use of L'Hospital's Theorem implies that when z approaches 1, the limit of the above rational function is a polynomial in ν of degree at most n.

Finally we prove (3). Since

$$\sharp(\nu\mathcal{P}) = \sum_{u\in H^2(BT)} \mathrm{DH}_{(\nu\mathcal{P})_+}(u) = \sum_{u\in H^2(BT)/\nu} \mathrm{DH}_{\mathcal{P}_+}(u),$$

it follows from the definition of the definite integral that

$$\lim_{\nu\to\infty} \frac{1}{\nu^n}\sharp(\nu\mathcal{P}) = \lim_{n\to\infty}\frac{1}{\nu^n} \sum_{u\in H^2(BT)/\nu} \mathrm{DH}_{\mathcal{P}_+}(u) = \int_{V^*} \mathrm{DH}_{\mathcal{P}}\, dV^* = \mathrm{vol}(\mathcal{P}),$$

proving that the coefficient of ν^n in $\sharp(\nu\mathcal{P})$ is $\mathrm{vol}(\mathcal{P})$.

We apply Lemma 7.3 to $0\mathcal{P}$, that is $\nu\mathcal{P}$ with $\nu = 0$. Then the u_I in the lemma is zero, and $\mathrm{DH}_{(0\mathcal{P})_+}(u) = \mathrm{WN}_{(0\mathcal{P})_+}(u) = 0$ unless $u = 0$ because the origin is the only vertex of $0\mathcal{P}$ so that the vertices of $(0\mathcal{P})_+$ are very close to the origin. Thus the right-hand side at the identity in the lemma applied to $0\mathcal{P}$ is a constant, say c, which is nothing but the constant term in $\sharp(\nu\mathcal{P})$. Now we approach z to ∞. Then the identity reduces to

$$\sum_{v\in C(I)} w(I) = c$$

because $\langle u_i^I, v\rangle > 0$ for all $i \in I$ if and only if $v = \sum_{i\in I} a_i v_i$ with $a_i > 0$ for all $i \in I$, and the latter is equivalent to saying that v belongs to the cone $C(I)$ spanned by v_i's ($i \in I$).

Since the left-hand side in the identity above is $\deg(\Delta)$ by definition, the constant term in $\sharp(\nu\mathcal{P})$, that is c, agrees with $\deg(\Delta)$. $\square$

Let N^*_Δ be the lattice of $N^*_\mathbb{R}$ generated by all u^I_i's for $I \in \Sigma^{(n)}$ and $i \in I$. If Δ is non-singular, then $N^*_\Delta = N^*$. The group ring $\mathbb{C}[N^*_\Delta]$ is a commutative $\mathbb{C}$-algebra, and it has a basis t^u ($u \in N^*_\Delta$) as a complex vector space with multiplication determined by the addition in N^*_Δ:

$$t^u \cdot t^{u'} := t^{u+u'}.$$

Each $v \in N$ such that $\langle u^I_i, v\rangle$ is an integer for any $I \in \Sigma^{(n)}$ and $i \in I$ determines a map from $\mathbb{C}[N^*_\Delta]$ to a Laurent polynomial ring $\mathbb{C}[z, z^{-1}]$ sending t^u to $z^{\langle u,v\rangle}$. Since Lemma 7.3 holds for any such v that $\langle u^I_i, v\rangle \neq 0$, we obtain

Corollary 7.4. *Let the notation be the same as in Lemma 7.3. Then*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{u_I}}{|N/N_I|} \sum_{g\in N/N_I} \frac{1}{\prod_{i\in I}(1-\chi_I(u^I_i, g)t^{-u^I_i})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u)t^u$$

as elements in the quotient ring of $\mathbb{C}[N^*_\Delta]$. *In particular, if the multi-fan* Δ *is non-singular, then* $N^*_\Delta = N^*$ *and*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{u_I}}{\prod_{i\in I}(1-t^{-u^I_i})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u)t^u.$$

For a later use, we shall rewrite $\chi_I(u^I_i, g)$. Consider a homomorphism $\eta\colon \mathbb{R}^d \to N_\mathbb{R}$ mapping $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ to $\sum_{i=1}^d a_i v_i \in N_\mathbb{R}$. For $I \in \Sigma^{(n)}$, we define

$$G'_I := \{\mathbf{a} \in \mathbb{R}^d \mid \eta(\mathbf{a}) \in N \text{ and } a_j = 0 \text{ for } j \notin I\}$$

and define G_I to be the projection image of G'_I on $\mathbb{R}^d/\mathbb{Z}^d$. Since v_i's ($i \in I$) are linearly independent and belong to N, G_I is a finite subgroup of $\mathbb{R}^d/\mathbb{Z}^d$ and η restricted to G'_I induces an isomorphism

$$\eta_I\colon G_I \cong N/N_I.$$

Note that $\eta_I([\mathbf{a}]) = [\sum_{i\in I} a_i v_i]$ where $[\ \ ]$ denotes the equivalence class.

On the other hand, for $i = 1, \dots, d$, let

$$\rho_i\colon \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}^*$$

be a homomorphism defined by $\rho_i([\mathbf{a}]) = \exp(2\pi\sqrt{-1}a_i)$.

Lemma 7.5. *For* $[\mathbf{a}] \in G_I \subset \mathbb{R}^d/\mathbb{Z}^d$ *and* $i \in I$, *we have* $\rho_i([\mathbf{a}]) = \chi_I(u^I_i, \eta_I([\mathbf{a}]))$.

Proof. Since $\eta_I([\mathbf{a}]) = [\sum_{i\in I} a_i v_i]$ and $\langle u^I_i, \sum_{i\in I} a_i v_i\rangle = a_i$, it follows from the definition (7.1) of χ_I that $\chi_I(u^I_i, \eta_I([\mathbf{a}])) = \exp(2\pi\sqrt{-1}a_i)$, which is equal to $\rho_i([\mathbf{a}])$ by definition. $\square$

Since G_I is isomorphic to N/N_I, Corollary 7.4 can be restated as follows.

Corollary 7.6. *Let the notation be as above. Then*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{u_I}}{|G_I|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\rho_i(g)t^{-u^I_i})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u)t^u$$

as elements in the quotient ring of $\mathbb{C}[N^*_\Delta]$.

8. Cohomological formula for $\sharp(\mathcal{P})$

Motivated by the geometrical observation done in subsequent sections 9 and 11, we define the "(equivariant) cohomology" of a complete simplicial multi-fan and the "(equivariant) first Chern class" of a multi-polytope. We then define an index map "in cohomology" and establish a "cohomological formula" describing $\sharp(\mathcal{P})$ for a lattice multi-polytope. This cohomological formula is a counterpart in combinatorics to the Hirzebruch-Riemann-Roch formula applied to a complex T-line bundle over a torus manifold. As an application of the cohomological formula, we show that the Khovanskii-Pukhlikov formula for a simple lattice convex polytope ([4] [5]) can be generalized to a simple lattice multi-polytope.

Let T be a compact torus of dimension $n = \operatorname{rank}_{\mathbb{Z}} N$ and let BT be the classifying space of T. Then $H_2(BT)$ is canonically isomorphic to $\operatorname{Hom}(S^1, T)$, the group consisting of homomorphisms from S^1 to T. In fact, a homomorphism $f\colon S^1 \to T$ induces a continuous map $Bf\colon BS^1 \to BT$ and once we fix a generator α of $H_2(BS^1) \cong \mathbb{Z}$, $(Bf)_*\alpha$ defines an element of $H_2(BT)$. The correspondence : $f \to (Bf)_*\alpha$ is known to be an isomorphism from $\operatorname{Hom}(S^1, T)$ to $H_2(BT)$. In the following we assume $N = H_2(BT)$ and identify it with $\operatorname{Hom}(S^1, T)$. Then $N^* = H^2(BT)$ is identified with $\operatorname{Hom}(T, S^1)$ and the group ring $\mathbb{C}[N^*]$ can be identified with the representation ring $R(T)$ of T.

Let $\Delta = (\Sigma, C, w^{\pm})$ be a complete simplicial multi-fan in N. Let $v_i \in H_2(BT)$ be a unique primitive vector in $C(\{i\})$ for each $i = 1, \dots, d$ as before. Motivated by the description of the equivariant cohomology of a compact non-singular toric variety (see Proposition 9.2 in the next section), we define $H_T^*(\Delta)$ to be the face ring of the augmented simplicial set Σ, i.e.,

$$H_T^*(\Delta) := \mathbb{Z}[x_1, \dots, x_d]/(x_I \mid I \notin \Sigma),$$

where $x_I = \prod_{i \in I} x_i$ and the degree of x_i is two, and call $H_T^*(\Delta)$ the *equivariant cohomology* of Δ. We also define a homomorphism $\pi^*\colon H^2(BT) \to H_T^2(\Delta)$ by

$$\pi^*(u) = \sum_{i=1}^{d} \langle u, v_i \rangle x_i, \tag{8.1}$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology. It extends to an algebra homomorphism $H^*(BT) \to H_T^*(\Delta)$, which we also denote by π^*. One can think of $H_T^*(\Delta)$ as a module (or more generally an algebra) over $H^*(BT)$ through π^*.

In the following we will mainly work with $\mathbb{Q}$ coefficients but the argument will work with $\mathbb{Z}$ coefficients when the multi-fan Δ is non-singular. Any homomorphism $f\colon A \to B$ between additive groups induces a homomorphism : $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ (or $A \otimes \mathbb{R} \to B \otimes \mathbb{R}$), which we also denote by f.

Lemma 8.1. *Any element in $H_T^*(\Delta) \otimes \mathbb{Q}$ can be written in the form $\sum_{J \in \Sigma} \pi^*(a_J) x_J$ with $a_J \in H^*(BT; \mathbb{Q})$ (not necessarily uniquely), in other words, $H_T^*(\Delta) \otimes \mathbb{Q}$ is generated by x_J's $(J \in \Sigma)$ as an $H^*(BT; \mathbb{Q})$-module.*

Proof. Let $\mathcal{I}$ denote a finite set which consists of elements in $\{1, \dots, d\}$ taken with multiplicity, i.e., elements in $\{1, \dots, d\}$ may appear in $\mathcal{I}$ repeatedly. Set $x_{\mathcal{I}} := \prod_{i \in \mathcal{I}} x_i$ and denote by $\bar{\mathcal{I}}$ the subset of $\{1, \dots, d\}$ consisting of elements appearing in $\mathcal{I}$. It follows from the definition that $H_T^*(\Delta)$ is additively generated by $x_{\mathcal{I}}$'s for $\mathcal{I}$ such that $\bar{\mathcal{I}} \in \Sigma$, so it suffices to prove the lemma for such $x_{\mathcal{I}}$. We shall prove it by induction on $[\mathcal{I}] := |\mathcal{I}| - |\bar{\mathcal{I}}|$.

If $[\mathcal{I}] = 0$, then $\mathcal{I} = \bar{\mathcal{I}} \in \Sigma$; so $x_{\mathcal{I}}$ is obviously of the form in the lemma in this case. Suppose $[\mathcal{I}] \geq 1$. Then there is an $i \in \mathcal{I}$ which appears in $\mathcal{I}$ at least twice. Set

$\mathcal{J} := \mathcal{I}\backslash\{i\}$. Then $\bar{\mathcal{J}} = \bar{\mathcal{I}} \in \Sigma$ and $[\mathcal{J}] = [\mathcal{I}] - 1$. Multiplying the both sides at (8.1) by $x_{\mathcal{J}}$, we obtain

$$\pi^*(u)x_{\mathcal{J}} = \langle u, v_i\rangle x_{\mathcal{I}} + \sum_{k\neq i}\langle u, v_k\rangle x_{\mathcal{J}\cup\{k\}}$$

for any $u \in H^2(BT;\mathbb{Q})$. We choose u such that $\langle u, v_i\rangle = 1$ and $\langle u, v_j\rangle = 0$ for all $j \in \mathcal{J}$ different from i. (Such u exists because $\{v_j \mid j \in \bar{\mathcal{J}}\}$ is a subset of a basis of $N_{\mathbb{Q}}$.) Then the identity above reduces to

$$x_{\mathcal{I}} = \pi^*(u)x_{\mathcal{J}} - \sum_{k\neq i, k\notin\mathcal{J}}\langle u, v_k\rangle x_{\mathcal{J}\cup\{k\}}.$$

Here $[\mathcal{J}\cup\{k\}] = [\mathcal{J}](= [\mathcal{I}] - 1)$ for $k \notin \mathcal{J}$, so the right-hand side above are of the form in the lemma by the induction assumption, showing that so is $x_{\mathcal{I}}$. This completes the induction step and proves the lemma. □

For $I \in \Sigma^{(n)}$, let $\{u_i^I \mid i \in I\}$ be the dual basis of $\{v_i \mid i \in I\}$ as before. We define a ring homomorphism $\iota_I^*\colon H_T^*(\Delta)\otimes\mathbb{Q} \to H^*(BT;\mathbb{Q})$ by

$$\iota_I^*(x_i) = \begin{cases} u_i^I & \text{if } i \in I, \\ 0 & \text{otherwise.}\end{cases}$$

This map is well-defined because x_J for $J \notin \Sigma$, which is zero in $H_T^*(\Delta)\otimes\mathbb{Q}$, maps to zero through ι_I^*.

Lemma 8.2. *The composition $\iota_I^*\circ\pi^*$ is the identity map, so ι_I^* is an $H^*(BT;\mathbb{Q})$-module map.*

Proof. Both π^* and ι_I^* are ring homomorphisms and $H^*(BT)$ is a polynomial ring generated by elements in $H^2(BT)$, so it suffices to check the lemma on $H^2(BT)$. Let $u \in H^2(BT)$. It follows from the definitions of π^* and ι_I^* that

$$(\iota_I^*\circ\pi^*)(u) = \iota_I^*(\sum_{i=1}^d\langle u, v_i\rangle x_i) = \sum_{i=1}^d\langle u, v_i\rangle u_i^I,$$

which agrees with u because $\{u_i^I \mid i \in I\}$ is the dual basis of $\{v_i \mid i \in I\}$. Since u is arbitrary, this proves that $\iota_I^*\circ\pi^*$ is the identity on $H^2(BT)$. □

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ is associated with real numbers c_i's by

$$\mathcal{F}(\{i\}) = \{u \in H^2(BT;\mathbb{R}) \mid \langle u, v_i\rangle = c_i\},$$

and these numbers determine an element $c_1^T(\mathcal{P}) := \sum_{i=1}^d c_ix_i$ of $H_T^2(\Delta)\otimes\mathbb{R}$, which we call the *equivariant first Chern class* of $\mathcal{P}$. This gives a bijective correspondence between the set of multi-polytopes defined on Δ and $H_T^2(\Delta)\otimes\mathbb{R}$. Note that $\iota_I^*(c_1^T(\mathcal{P}))$ agrees with the vertex $\cap_{i\in I}\mathcal{F}(\{i\})$. If $\mathcal{P}$ is a lattice multi-polytope (i.e., the vertices of $\mathcal{P}$ lie in the lattice $H^2(BT)$ of $H^2(BT;\mathbb{R})$), then c_i's are integers and the u_I in Corollary 7.4 or 7.6 agrees with $\iota_I^*(c_1^T(\mathcal{P}))$. When Δ is non-singular, $\mathcal{P}$ is a lattice multi-polytope if and only if the c_i's are all integers, but otherwise the "if" part does not hold, in other words, an element of $H_T^2(\Delta)$ is not necessarily realized as the equivariant first Chern class of a lattice multi-polytope. However, there is a nonzero integer m such that mx for any $x \in H_T^2(\Delta)$ is realized as the equivariant first Chern class of a lattice multi-polytope because $|N/N_I|\iota_I^*(x)$'s lie in $H^2(BT)$.

Lemma 8.3. *For any $J \in \Sigma$, we have*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)\iota_I^*(\prod_{j\in J}(e^{mx_j}-1))}{|G_I|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\rho_i(g)e^{-u_i^I})} \in H^{**}(BT;\mathbb{Q}).$$

where $H^{**}(BT;\mathbb{Q}) := \prod_{q=0}^{\infty} H^q(BT;\mathbb{Q})$.

Proof. Since $\prod_{j\in J}(e^{mx_j}-1)$ is a linear combination of $\prod_{k\in K} e^{mx_k} = e^{m\sum_{k\in K} x_k}$ for $K \in \Sigma$, it suffices to show that

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)\iota_I^*(e^{m\sum x_k})}{|G_I|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\rho_i(g)e^{-u_i^I})} \in H^{**}(BT;\mathbb{Q}). \tag{8.2}$$

As remarked above, $m\sum_{k\in K} x_k$ is realized as the equivariant first Chern class of a lattice multi-polytope, so it follows from Corollary 7.6 that

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{\iota_I^*(m\sum x_k)}}{|G_I|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\rho_i(g)t^{-u_i^I})} \in \mathbb{C}[N^*] = R(T).$$

The Chern character : $\mathbb{C}[N^*] = R(T) \to H^{**}(BT;\mathbb{Q})$ mapping t^u to e^u extends to a map from $\mathbb{C}[N^*_\Delta]$ and it further extends to a map between their quotient rings. Sending the element above by this extended Chern character, we obtain (8.2). □

Let S be the multiplicative set consisting of nonzero homogeneous elements of positive degree in $H^*(BT;\mathbb{Q})$. Since $H^*(BT;\mathbb{Q})$ is a polynomial ring (hence an integral domain), $H^*(BT;\mathbb{Q})$ can be thought of as a subring of the localized ring $S^{-1}H^*(BT;\mathbb{Q})$. We define the index map

$$\pi_!\colon H_T^*(\Delta)\otimes\mathbb{Q} \to S^{-1}H^*(BT;\mathbb{Q})$$

"in cohomology" by

$$\pi_!(x) := \sum_{I\in\Sigma^{(n)}} \frac{w(I)\iota_I^*(x)}{|G_I|\prod_{i\in I} u_i^I}$$

(cf. [2, (3.8)]). This map decreases degrees by $2n$, and is an $H^*(BT;\mathbb{Q})$-module map by Lemma 8.2.

Lemma 8.4. *The image of $\pi_!$ lies in $H^*(BT;\mathbb{Q})$.*

Proof. Since $\pi_!$ is an $H^*(BT;\mathbb{Q})$-module map, it suffices to check the lemma for elements x_J's ($J \in \Sigma$) by Lemma 8.1. We distinguish two cases.

Case 1. The case where $|J| = n$, i.e., $J \in \Sigma^{(n)}$. In this case

$$\iota_I^*(x_J) = \begin{cases} \prod_{i\in I} u_i^I & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\pi_!(x_J) = \sum_{I\in\Sigma^{(n)}} \frac{w(I)\iota_I^*(x_J)}{|G_I|\prod_{i\in I} u_i^I} = \frac{w(J)}{|G_J|} \in H^0(BT;\mathbb{Q}).$$

Case 2. The case where $|J| < n$. In this case we will show that $\pi_!(x_J) = 0$. Since $\rho_i(g) = 1$ for any $i \in I$ if and only if g is the identity, and

$$\prod_{i\in I}(1 - e^{-u_i^I}) = (\prod_{i\in I} u_i^I)(1 + \text{higher degree term})$$
$$\prod_{j\in J}(e^{mx_j} - 1) = m^{|J|}x_J(1 + \text{higher degree term}),$$

the term of lowest degree in Lemma 8.3 (up to a nonzero constant multiple) is

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)\iota_I^*(x_J)}{|G_I|\prod_{i\in I} u_i^I},$$

that is, $\pi_!(x_J)$, and Lemma 8.3 tells us that it is an element of $H^*(BT;\mathbb{Q})$. This means that $\pi_!(x_J) = 0$ because $\pi_!$ decreases degrees by $2n$ and $|J| < n$. □

Now, motivated by the description of the cohomology ring of a compact non-singular toric variety (see p.106 in [9]), we define $H^*(\Delta)$ to be the quotient ring of $H_T^*(\Delta)$ by the ideal generated by $\pi^*(H^2(BT))$, in other words,

$$H^*(\Delta) := \mathbb{Z}[x_1,\dots,x_d]/\mathfrak{A},$$

where $\mathfrak{A}$ is the ideal generated by all

(1) x_I for $I \notin \Sigma$,
(2) $\sum_{i=1}^d \langle u, v_i\rangle x_i$ for $u \in N$.

Since $\pi_!$ is an $H^*(BT;\mathbb{Q})$-module map and $H^*(BT;\mathbb{Q})/(H^2(BT;\mathbb{Q}))$ is isomorphic to $H^0(BT;\mathbb{Q}) = \mathbb{Q}$, $\pi_!$ induces a homomorphism

$$\int_\Delta : H^*(\Delta)\otimes\mathbb{Q} \to \mathbb{Q},$$

where only elements of degree $2n$ in $H^*(\Delta)\otimes\mathbb{Q}$ survive through the map $\int_\Delta$.

Remember that G_I is a finite subgroup of $\mathbb{R}^d/\mathbb{Z}^d$. We denote by G_Δ the union of G_I over all $I \in \Sigma^{(n)}$. Since ρ_i is defined on $\mathbb{R}^d/\mathbb{Z}^d$, $\rho_i(g)$ makes sense for $g \in G_\Delta$. It follows from the definition of G_I and ρ_i that if $g \in G_I$, then $\rho_i(g) = 1$ for $i \notin I$.

We define the *equivariant Todd class* $\mathcal{T}^T(\Delta)$ of the complete simplicial multi-fan Δ by

$$\mathcal{T}^T(\Delta) := \sum_{g\in G_\Delta}\prod_{i=1}^d \frac{x_i}{1-\rho_i(g)e^{-x_i}} \in H_T^{**}(\Delta)\otimes\mathbb{Q},$$

and the *Todd class* $\mathcal{T}(\Delta)$ of Δ by

$$\mathcal{T}(\Delta) := \sum_{g\in G_\Delta}\prod_{i=1}^d \frac{\bar{x}_i}{1-\rho_i(g)e^{-\bar{x}_i}} \in H^{**}(\Delta)\otimes\mathbb{Q},$$

where $\bar{x}_i$ denotes the image of $x_i \in H_T^*(\Delta)$ in $H^*(\Delta)$ (cf. [5]). We also define the *first Chern class* $c_1(\mathcal{P})$ of a multi-polytope $\mathcal{P}$ defined on Δ to be the image of $c_1^T(\mathcal{P}) \in H_T^2(\Delta)\otimes\mathbb{R}$ in $H^2(\Delta)\otimes\mathbb{R}$.

Theorem 8.5. *If $\mathcal{P}$ is a simple lattice multi-polytope, then* $\int_\Delta e^{c_1(\mathcal{P})}\mathcal{T}(\Delta) = \sharp(\mathcal{P})$.

Proof. We shall compute $\pi_!(e^{c_1^T(\mathcal{P})}\mathcal{T}^T(\Delta))$. For that, we need to see $\iota_I^*(\mathcal{T}^T(\Delta))$. Let $g \in G_\Delta$. If $g \notin G_I$, then there is an $i \notin I$ such that $\rho_i(g) \neq 1$; so

$$\iota_I^*\Big(\frac{x_i}{1-\rho_i(g)e^{-x_i}}\Big) = 0$$

for such i because the Maclaurin expansion of $x_i/(1-\rho_i(g)e^{-x_i})$ has no constant term and $\iota_I^*(x_i) = 0$. Therefore, only elements g in G_I contribute to $\iota_I^*(\mathcal{T}^T(\Delta))$. Now suppose $g \in G_I$. Then $\rho_i(g) = 1$ for $i \notin I$, so

$$\iota_I^*\Big(\frac{x_i}{1-\rho_i(g)e^{-x_i}}\Big) = 1$$

for such i because the Maclaurin expansion of $x_i/(1-\rho_i(g)e^{-x_i})$ has the constant term 1 and $\iota_I^*(x_i) = 0$. Finally, since $\iota_I^*(x_i) = u_i^I$ for $i \in I$, we thus have

$$\iota_I^*(\mathcal{T}^T(\Delta)) = \sum_{g\in G_I}\prod_{i\in I}\frac{u_i^I}{1-\rho_i(g)e^{-u_i^I}}.$$

This together with the definition of $\pi_!$ and Corollary 7.6 shows that

$$\begin{aligned}\pi_!(e^{c_1^T(\mathcal{P})}\mathcal{T}^T(\Delta)) =& \pi_!\Big(e^{c_1^T(\mathcal{P})}\sum_{g\in G_\Delta}\prod_{i=1}^{d}\frac{x_i}{1-\rho_i(g)e^{-x_i}}\Big)\\ =& \sum_{I\in\Sigma^{(n)}}\frac{w(I)e^{\iota_I^*(c_1^T(\mathcal{P}))}}{|G_I|}\sum_{g\in G_I}\frac{1}{\prod_{i\in I}(1-\rho_i(g)e^{-u_i^I})}\\ =& \sum_{u\in H^2(BT)}\mathrm{DH}_{\mathcal{P}_+}(u)e^u.\end{aligned}$$

This implies that

$$\int_\Delta e^{c_1(\mathcal{P})}\mathcal{T}(\Delta) = \sum_{u\in H^2(BT)}\mathrm{DH}_{\mathcal{P}_+}(u) = \sharp(\mathcal{P}).$$

□

Remark. The argument developed above in this section is purely combinatorial, but it is possible to take a topological approach. Namely, associated with a complete simplicial multi-fan Δ, one can construct a torus space M_Δ with $H_T^*(M_\Delta;\mathbb{Q}) = H_T^*(\Delta)\otimes\mathbb{Q}$ (see [6]). It is not necessarily a manifold but has a fundamental class so that the equivariant Gysin homomorphism $\pi_!\colon H_T^*(M_\Delta;\mathbb{Q}) = H_T^*(\Delta)\otimes\mathbb{Q} \to H_T^{*-2n}(pt;\mathbb{Q}) = H^{*-2n}(BT;\mathbb{Q})$, that is, the index map, can be defined.

As an application of the theorem above, we shall show that Khovanskii-Pukhlikov formula, which relates a certain variation of the volume of a simple convex lattice polytope to the number of lattice points in it, can be generalized to simple multi-polytopes. We begin with

Lemma 8.6. $\mathrm{vol}(\mathcal{P}) = \frac{1}{n!}\int_\Delta c_1(\mathcal{P})^n = \int_\Delta e^{c_1(\mathcal{P})}$ *for a simple multi-polytope* $\mathcal{P}$.

Proof. The latter identity is obvious because only elements of degree $2n$ in $H^*(\Delta)\otimes\mathbb{R}$ survive through the map $\int_\Delta$. We shall prove the former identity.

Step 1. If $\mathcal{P}$ is a lattice multi-polytope, then Theorem 8.5 applied to $\nu\mathcal{P}$ for any integer ν implies

$$\int_\Delta e^{c_1(\nu\mathcal{P})}\mathcal{T}(\Delta) = \sharp(\nu\mathcal{P}).$$

We compare the coefficients of ν^n at the both sides above. Since $c_1(\nu\mathcal{P}) = \nu c_1(\mathcal{P})$, the coefficient of ν^n at the left-hand side is $\frac{1}{n!}\int_\Delta c_1(\mathcal{P})^n$, while the one at the right-hand side is $\operatorname{vol}(\mathcal{P})$ by Theorem 7.2 (3). Therefore the lemma is proven for a lattice multi-polytope $\mathcal{P}$.

Step 2. If $\mathcal{P}$ is *rational*, by which we mean that there is a nonzero integer m such that $m\mathcal{P}$ is a lattice multi-polytope, then $\operatorname{vol}(m\mathcal{P}) = \frac{1}{n!}\int_\Delta c_1(m\mathcal{P})^n$ by Step 1. Since $\operatorname{vol}(m\mathcal{P}) = m^n \operatorname{vol}(\mathcal{P})$ and $c_1(m\mathcal{P}) = mc_1(\mathcal{P})$, the lemma is proven for a rational multi-polytope $\mathcal{P}$.

Step 3. The functions $\operatorname{vol}(\cdot)$ and $\int_\Delta c_1(\cdot)^n$ are defined on the vector space $H^2_T(\Delta)\otimes\mathbb{R}$ through the equivariant first Chern class, and they are obviously continuous. By Step 2 they agree on all rational multi-polytopes which form a dense subset of the vector space, so they must agree on the entire vector space by continuity. This completes the proof of the lemma. □

Multi-polytopes defined on Δ form a vector space isomorphic to $H^2_T(\Delta)\otimes\mathbb{R}$ through the equivariant first Chern class, and Lemma 8.6 implies that the volume function is a homogeneous polynomial function of degree n. In fact, if one writes $c_1^T(\mathcal{P}) = \sum_{i=1}^d c_i x_i$, then $\operatorname{vol}(\mathcal{P})$ is a homogeneous polynomial in $c_1,\dots,c_d$ of degree n.

For $h = (h_1,\dots,h_d)\in\mathbb{R}^d$, we denote by $\mathcal{P}_h$ a multi-polytope with $c_1^T(\mathcal{P}_h) = \sum_{i=1}^d (c_i + h_i)x_i$. Since $c_1(\mathcal{P}_h) = \sum_{i=1}^d (c_i+h_i)\bar{x}_i$, Lemma 8.6 applied to $\mathcal{P}_h$ implies that $\operatorname{vol}(\mathcal{P}_h)$ is a polynomial in $h_1,\dots,h_d$ (of total degree n). We define the *Todd operator* as follows:

$$\mathcal{T}(\partial/\partial h) := \sum_{g\in G_\Delta}\prod_{i=1}^d \frac{\partial/\partial h_i}{1-\rho_i(g)e^{-\partial/\partial h_i}}.$$

Although the Todd operator is of infinite order, its operation on $\operatorname{vol}(\mathcal{P}_h)$ converges because $\operatorname{vol}(\mathcal{P}_h)$ is a polynomial in $h_1,\dots,h_d$. The following theorem extends the Khovanskii-Pukhlikov formula [21] [4] [5] to simple lattice multi-polytopes.

Theorem 8.7. *If $\mathcal{P}$ is a simple lattice multi-polytope, then*

$$\mathcal{T}(\partial/\partial h)\operatorname{vol}(\mathcal{P}_h)|_{h=0} = \sharp(\mathcal{P}).$$

Proof. An elementary computation shows that

$$\frac{\partial/\partial h_i}{1-\rho_i(g)e^{-\partial/\partial h_i}}e^{(c_i+h_i)\bar{x}_i}|_{h_i=0} = e^{c_i\bar{x}_i}\frac{\bar{x}_i}{1-\rho_i(g)e^{-\bar{x}_i}}.$$

Therefore, it follows from Lemma 8.6 and Theorem 8.5 that

$$\begin{aligned}\mathcal{T}(\partial/\partial h)\operatorname{vol}(\mathcal{P}_h)|_{h=0} &= \mathcal{T}(\partial/\partial h)\int_\Delta e^{c_1(\mathcal{P}_h)}|_{h=0} = \int_\Delta \mathcal{T}(\partial/\partial h)e^{c_1(\mathcal{P}_h)}|_{h=0}\\ &= \int_\Delta \sum_{g\in G_\Delta}\prod_{i=1}^d \frac{\partial/\partial h_i}{1-\rho_i(g)e^{-\partial/\partial h_i}}e^{(c_i+h_i)\bar{x}_i}|_{h_i=0}\\ &= \int_\Delta \sum_{g\in G_\Delta}\prod_{i=1}^d e^{c_i\bar{x}_i}\frac{\bar{x}_i}{1-\rho_i(g)e^{-\bar{x}_i}}\\ &= \int_\Delta e^{c_1(\mathcal{P})}\mathcal{T}(\Delta) = \sharp(\mathcal{P}),\end{aligned}$$

proving the theorem. □

Remark. One can reformulate the Khovanskii-Pukhlikov formula as follows. As remarked above, the volume function vol is a polynomial in $c_1,\dots,c_d$, so one can apply the Todd operator $\mathcal{T}(\partial/\partial c)$ (with the variables $c=(c_1,\dots,c_d)$ instead of $h=(h_1,\dots,h_d)$) to the volume function vol and evaluate at a simple lattice multi-polytope $\mathcal{P}$. The same argument as in the proof of Theorem 8.7 shows that the evaluated value agrees with $\sharp(\mathcal{P})$.

9. Multi-fan of a torus manifold

In this section we introduce the notion of a torus manifold and associate a complete non-singular multi-fan with it. A compact non-singular toric variety provides an example of a torus manifold, but the class of torus manifolds is much wider than that of compact non-singular toric varieties, (apparently, even wider than that of unitary toric manifolds introduced in [23]). The basic theory of toric varieties says that there is a one-to-one correspondence between compact non-singular toric varieties and complete non-singular fans. This correspondence is extended in one direction, namely from torus manifolds to complete non-singular multi-fans. But the usual way to associate a fan with a toric variety (see [9, Section 2.3]) does not work in our extended category. However, when a toric variety is compact and non-singular, the corresponding (complete and non-singular) fan can be reproduced using equivariant cohomology and this argument works even for torus manifolds. The idea is essentially same as in [23].

We begin with the definition of a torus manifold. An elementary representation theory of a torus group tells us that if an m-dimensional torus $(S^1)^m$ acts effectively and smoothly on a connected smooth manifold of dimension $2n$ with non-empty fixed point set, then $m\le n$ and the dimension of the fixed point set is at most $2(n-m)$. We are interested in an extreme case $m=n$. Let M be a closed, connected, smooth manifold of dimension $2n$ with an effective smooth action of an n-dimensional torus group $T=(S^1)^n$ such that the fixed point set M^T is non-empty. Then M^T is necessarily isolated. A closed, connected, codimension two submanifold of M is called *characteristic* if it is a connected component of the set fixed pointwise by a certain circle subgroup of T and contains at least one T-fixed point. Since M is compact, there are only finitely many characteristic submanifolds. We denote them by M_i $(i=1,\dots,d)$. They are orientable if M is orientable.

Definition. Let M be a closed, connected, oriented, smooth manifold M of dimension $2n$ with an effective smooth action of an n-dimensional torus group T with non-empty fixed point set M^T. M will be called a *torus manifold* if a prefered orientation is given for each characteristic submanifold M_i.

A toric variety X (of dimension n) is a normal complex algebraic variety of complex dimension n with an effective algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit. If X is compact and non-singular, then X with the restricted action of T $(\subset (\mathbb{C}^*)^n)$ provides an example of a torus manifold of dimension $2n$. In this case, characteristic submanifolds are $(\mathbb{C}^*)^n$-invariant divisors. They have canonical orientations since they are complex manifolds. Similarly, when a torus manifold is equipped with a T-invariant unitary structure, characteristic submanifolds have canonical orientations. With these orientations of characteristic submanifolds, the torus manifold will be called a *unitary torus manifold* (also called a unitary toric manifold in [23]).

Example 9.1. A complex projective space $\mathbb{C}P^n$ with an action of $(\mathbb{C}^*)^n$ given by

$$[z_0, z_1, \dots, z_n] \to [z_0, g_1 z_1, \dots, g_n z_n],$$

where $[z_0, z_1, \dots, z_n] \in \mathbb{C}P^n$ and $(g_1, \dots, g_n) \in (\mathbb{C}^*)^n$, is a compact non-singular toric variety. This with the restricted T-action is a torus manifold and there are $n+1$ characteristic submanifolds, that are respectively defined by $z_i = 0$ for $i = 0, 1, \dots, n$.

There are many torus manifolds which do not arise from compact non-singular toric varieties, see [6], [23], [26].

Henceforth M will denote a torus manifold of dimension $2n$. Let $p \in M^T$. Since M^T is isolated, the tangential T-representation $\tau_p M$ of M at p has no trivial factor, so it decomposes into the direct sum of n irreducible real two-dimensional T-representations. This implies that there are exactly n characteristic submanifolds which contain p. In fact, an irreducible factor in $\tau_p M$ corresponds to the normal direction to a characteristic submanifold at p. We set

$$\Sigma(M) := \{I \subset \{1, \dots, d\} \mid (\cap_{i \in I} M_i)^T \neq \phi\}.$$

We add an empty set to $\Sigma(M)$ as a member, so that $\Sigma(M)$ becomes an augmented simplicial set. The observation above implies that the cardinality of an element in $\Sigma(M)$ is at most n and there is an element in $\Sigma(M)$ with cardinality n.

The augmented simplicial set $\Sigma(M)$ is closely related to the ring structure of the equivariant cohomology $H^*_T(M)$ of M with integer coefficients. Let us explain this briefly. Since M_i and M are oriented closed T-manifolds and the codimension of M_i is two, the inclusion map from M_i to M induces a Gysin homomorphism $H^*_T(M_i) \to H^{*+2}_T(M)$ in equivariant cohomology which raises dgrees by two (see [19] for example). Denote by $\xi_i \in H^2_T(M)$ the image of the identity element in $H^0_T(M_i)$. We may think of ξ_i as the Poincaré dual of M_i (considered as a cycle in M) in equivariant cohomology. If the orientation on M or M_i is reversed, then ξ_i turns into $-\xi_i$.

We take a polynomial ring $\mathbb{Z}[x_1, \dots, x_d]$ in d-variables and consider a map

$$\varphi\colon \mathbb{Z}[x_1, \dots, x_d] \to H^*_T(M)$$

which sends x_i to ξ_i. This map is often surjective. Here is a case.

Proposition 9.2. ([23], Proposition 3.4.) *Suppose that $H^*(M)$ is generated by elements in $H^2(M)$ as a ring (this is the case when M is a compact non-singular toric variety). Then the map φ is surjective and the kernel is the ideal generated by monomials $\prod_{i \in I} x_i$ for all subsets $I \subset \{1, \dots, d\}$ such that $I \notin \Sigma(M)$. In other words, $H^*_T(M)$ is isomorphic to the face ring (or Stanley-Reisner ring) of $\Sigma(M)$.*

The equivariant cohomology $H^*_T(M)$ has a finer structure than the ring structure. The map π collapsing M to a point induces a homomorphism $\pi^*\colon H^*_T(pt) = H^*(BT) \to H^*_T(M)$, so that $H^*_T(M)$ can be viewed as an algebra over $H^*(BT)$ through π^*. This algebra structure over $H^*(BT)$ cannot be determined by $\Sigma(M)$ and contains more information on the torus manifold M. To see the algebra structure, it is enough to see the image of $H^2(BT)$ by π^* because $H^*(BT)$ is a polynomial ring generated by elements in $H^2(BT)$.

Lemma 9.3. ([23], Lemma 1.5.) *For each $i \in \{1, \dots, d\}$ there exists a unique element $v_i \in H_2(BT)$ such that*

$$\pi^*(u) = \sum_{i=1}^{d} \langle u, v_i \rangle \xi_i \qquad \textit{modulo } H^*(BT)\textit{-torsions}$$

for any $u \in H^2(BT)$, where $\langle\ ,\ \rangle$ denotes the usual pairing between cohomology and homology.

Proof. The proof is given in [23], but we shall give a simple proof for the reader's convenience when M is as in Proposition 9.2. Since $H^2_T(M)$ is additively generated by ξ_i's, one can express

$$\pi^*(u) = \sum_{i=1}^{d} v_i(u)\xi_i$$

with a unique integer $v_i(u)$ depending on u for each i. We view $v_i(u)$ as a function of $u \in H^2(BT)$. Since it is linear, it defines an element v_i of $\mathrm{Hom}(H^2(BT), \mathbb{Z}) = H_2(BT)$ such that $v_i(u) = \langle u, v_i \rangle$. □

Note. A geometrical interpretation of the vectors v_i will be given in Section 12.

In order to introduce a multi-fan, we adopt $H_2(BT)$ as the lattice N and identify $H_2(BT; \mathbb{R})$ with the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Then we define a map

$$C(M)\colon \Sigma(M) \to \mathrm{Cone}(N)$$

by sending $I \in \Sigma(M)$ to the cone in $H_2(BT; \mathbb{R})$ spanned by v_i's ($i \in I$) (and the empty set to $\{0\}$).

Finally we shall define a pair of weight functions on maximal cones of dimension n. Remember that a characteristic submanifold M_i is a connected component of the set fixed pointwise by a certain circle subgroup, say T_i, of T. It turns out that T_i agrees with the circle subgroup determined by $v_i \in H_2(BT)$ through the natural identification $H_2(BT) \cong \mathrm{Hom}(S^1, T)$ ([23], Lemma 1.10). Therefore $M_I := \cap_{i \in I} M_i$ is fixed pointwise by a subtorus T_I generated by T_i's for $i \in I$.

Lemma 9.4. ([23], Lemma 1.7.) *Suppose $I \in \Sigma(M)^{(n)}$. Then the set $\{v_i \mid i \in I\}$ forms a basis of $H_2(BT)$, so that M_I is a subset of M^T and the cone $C(M)(I)$ is of dimension n.*

A fixed point $p \in M^T$ belongs to M_I for some $I \in \Sigma^{(n)}$, and the tangent space $\tau_p M$ at $p \in M_I$ naturally decomposes into

$$\tau_p M \cong \bigoplus_{i \in I} (\tau_p M / \tau_p M_i).$$

The orientations on M and M_i determine an orientation on τ_pM/τ_pM_i for each i, and then an orientation on τ_pM through the above isomorphism. On the other hand, τ_pM has a given orientation since M is oriented. These two orientations on τ_pM may disagree. We define the sign ϵ_p at p to be $+1$ or -1 according as the two orientations agree or disagree, and set

$$w(M)^+(I) := \text{the number of } \{p \in M_I \mid \epsilon_p = +1\},$$
$$w(M)^-(I) := \text{the number of } \{p \in M_I \mid \epsilon_p = -1\}.$$

Note that $w(M)^+(I) = 1$ and $w(M)^-(I) = 0$ for all $I \in \Sigma^{(n)}$ if M is a compact non-singular toric variety.

Definition. We call the triple $\Delta(M) := (\Sigma(M), C(M), w(M)^{\pm})$ the multi-fan of M.

A characteristic submanifold of M_i is a connected component of $M_i \cap M_j$ for some j containing a T-fixed point. We give it the orientation induced from those on M_i and M_j. With these orientations equipped, M_i, on which T/T_i acts effectively, is considered as a torus manifold. If $M_i \cap M_j$ is connected for any $j \in \Sigma(M)_i^{(1)}$ (this is the case when M is a compact non-singular toric variety), then the multi-fan $\Delta(M_i)$ of M_i agrees with the projected multi-fan $\Delta(M)_i$ with respect to $\{i\} \in \Sigma(M)^{(1)}$. They are different otherwise but there is a natural surjective map from $\Sigma(M_i)$ to $\Sigma(M)_i$.

Similarly, a connected component of M_K for $K \in \Sigma(M)$ containing a T-fixed point is considered as a torus manifold, and $\Delta(M_K)$ agrees with $\Delta(M)_K$ if M_K and $M_K \cap M_j$ are connected for all $j \in \Sigma(M)_K^{(1)}$, but otherwise they are different although there is a natural surjective map from $\Sigma(M_K)$ to $\Sigma(M)_K$, where $\Sigma(M_K)$ is an augmented simplicial set obtained from the union of the simplicial sets associated with the connected components of M_K.

The multi-fan $\Delta(M)$ is non-singular by Lemma 9.4. We shall show that it is complete.

Lemma 9.5. *$\Delta(M)$ is complete.*

Proof. As we remarked in Section 2 after the definition of the completeness of a multi-fan, it suffices to prove the pre-completeness of $\Delta(M)_J$ for any $J \in \Sigma(M)^{(n-1)}$. Choose a generic vector v from $N = H_2(BT)$. The sign $(-1)^{\{i\}}$ for $i \in \Sigma(M)_J^{(1)}$ is defined as in Section 5 with respect to the projection image of v on the quotient lattice of N by the sublattice generated by $C(M)(J) \cap N$. The pre-completeness of $\Delta(M)_J$ is equivalent to this identity:

$$\sum_{\{i\} \in \Sigma(M)_J^{(1)}} (-1)^{\{i\}} w(M)_J(\{i\}) = 0,$$

which we will verify in the following. Since $|J| = n-1$, a connected component of M_J containing a T-fixed point is a 2-dimensional sphere on which $T^J := T/T_J$ acts effectively. We denote those connected components by S^2_α's. They are torus manifolds equipped with the orientations discussed before this lemma. Since S^2_α has two T^J-fixed points, $\Sigma(S^2_\alpha)^{(1)}$ consists of two elements, denoted by $\alpha_\pm$, corresponding to the T^J-fixed points. One easily checks that the multi-fan $\Delta(S^2_\alpha)$ of S^2_α is complete, which is equivalent to this identity:

$$(-1)^{\alpha_+} w(S^2_\alpha)(\alpha_+) + (-1)^{\alpha_-} w(S^2_\alpha)(\alpha_-) = 0. \tag{9.1}$$

As discussed before this lemma, we have a natural map $\pi_J\colon \Sigma(M_J) \to \Sigma(M)_J$. Note that if $\pi_J(\alpha_\epsilon) = \{i\}$ where ϵ stands for $+$ or $-$, then $(-1)^{\alpha_\epsilon} = (-1)^{\{i\}}$. On the other hand, we have

$$w(M)_J(\{i\}) = \sum_{\pi_J(\alpha_\epsilon)=\{i\}} w(S^2_\alpha)(\alpha_\epsilon).$$

Therefore

$$\sum_{\{i\}\in\Sigma(M)^{(1)}_J} (-1)^{\{i\}} w(M)_J(\{i\}) = \sum_{\alpha_\epsilon} (-1)^{\alpha_\epsilon} w(S^2_\alpha)(\alpha_\epsilon),$$

which vanishes by (9.1), proving the lemma. □

We make a remark on orientations at this point. Choose an orientation on T and fix it. It induces an orientation on $H_2(BT;\mathbb{R})$, so that $[\Delta(M)] \in H_{n-1}(\Sigma(M))$ is defined. If the orientation on T or M is reversed, then $[\Delta(M)]$ turns into $-[\Delta(M)]$. But we have

Lemma 9.6. $[\Delta(M)]$ *does not depend on the orientations on* M_i*'s.*

Proof. Recall that the cycle which defines $[\Delta(M)]$ is $\sum_{I\in\Sigma(M)^{(n)}} w(M)(I)\langle I\rangle$. We reverse the orientation on M_i. Obviously, $w(M)(I)$ and $\langle I\rangle$ remain unchanged unless $i \in I$. Suppose $i \in I$. Then, since the orientation on $\tau_p M/\tau_p M_i$ is reversed, $w(M)^+(I)$ and $w(M)^-(I)$ will be interchanged, so that $w(M)(I)$ turns into $-w(M)(I)$. As for $\langle I\rangle$, ξ_i turns into $-\xi_i$ as remarked before and hence so does v_i by Lemma 9.3. Thus, $\langle I\rangle$ turns into $-\langle I\rangle$ if $i \in I$. After all, $w(M)(I)\langle I\rangle$ does not depend on the orientations on M_i's for any $I \in \Sigma(M)^{(n)}$. □

Remember that there is a canonical isomorphism $\mathrm{Hom}(T,S^1) \cong H^2(BT)$. We denote by t^u the element in $\mathrm{Hom}(T,S^1)$ corresponding to $u \in H^2(BT)$. Elements of $\mathrm{Hom}(T,S^1)$ are complex one-dimensional representations of T and they generate the representation ring $R(T)$ of T which is identified with the group ring of $H^2(BT)$. Since ξ_i is the image of $1 \in H^0_T(M_i)$ by the equivariant Gysin map from M_i to M, its restriction to a T-fixed point p in M_i, denoted by $\xi_i|_p$, gives the equivariant Euler class of the T-representation $\tau_p M/\tau_p M_i$; so $\tau_p M/\tau_p M_i = t^{\xi_i|_p}$. On the other hand, the identity in Lemma 9.3 restricted to p shows that $\{\xi_i|_p \mid i \in I\}$ is the dual basis of $\{v_i \mid i \in I\}$, so $\xi_i|_p$ is independent of the choice of $p \in M_I$ and $\xi_i|_p = u^I_i$ in the notation of Section 7. Therefore we have

$$\tau_p M = \bigoplus_{i\in I} t^{u^I_i}$$

as a T-representation whenever $p \in M_I$.

The elements ξ_i's ($i = 1,\dots,d$) generate $H^2_T(M)$ additively modulo $H^*(BT)$-torsions ([23, Lemma 3.2]) and the torsion elements vanish when restricted to the fixed point set M^T because $H^*_T(M^T)$ is a free $H^*(BT)$-module. Since the restriction $\xi_i|_p$ ($p \in M_I$) depends on only I, we shall denote an element $\xi \in H^2_T(M)$ restricted to a point in M_I by $\xi|_I$. Note that

$$\xi_i|_I = \begin{cases} u^I_i & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

Lemma 9.7. *For any* $\xi \in H^2_T(M)$,

$$\sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)t^{\xi|_I}}{\prod_{i\in I}(1-t^{-u^I_i})}$$

is an element of $R(T)$ *when* M *is a torus manifold.*

Proof. Since ξ_i's generate $H^2_T(M)$ additively modulo $H^*(BT)$-torsions, $\xi = \sum_{i=1}^d c_i\xi_i$ modulo $H^*(BT)$-torsions with some integers c_i's. We define a map $\mathcal{F}_\xi\colon \Sigma(M)^{(1)} \to \mathrm{HP}(H^2(BT;\mathbb{R}))$ by

$$\mathcal{F}_\xi(\{i\}) := \{u \in H^2(BT;\mathbb{R}) \mid \langle u, v_i\rangle = c_i\}.$$

The pair $(\Delta(M), \mathcal{F}_\xi)$ is a lattice multi-polytope, and $\cap_{i\in I}\mathcal{F}_\xi(\{i\}) = \xi|_I$ for $I \in \Sigma(M)^{(n)}$ which follows from (9.2). Since $\Delta(M)$ is non-singular by Lemma 9.4 and complete by Lemma 9.5, the lemma follows from Corollary 7.4 applied to the multi-polytope $(\Delta(M), \mathcal{F}_\xi)$. □

10. T_y-genus of a torus manifold

When M is a unitary torus manifold, the localization formula of the T_y-genus $T_y[M]$ of M tells us that

$$T_y[M] = \sum_{I\in\Sigma^{(n)}} w(M)(I)\frac{\prod_{i\in I}(1+yt^{-u_i^I})}{\prod_{i\in I}(1-t^{-u_i^I})} \tag{10.1}$$

and this is actually a polynomial in y with constant coefficients. As is well known, $T_0[M]$ agrees with the Todd genus of M and $T_1[M]$ agrees with the signature of M, see [17]. The T_y-genus is a genus for unitary manifolds and it is not defined for general torus manifolds. But the right-hand side of (10.1) makes sense even for a torus manifold, and we take it as the definition of the T_y-genus $T_y[M]$ of M and define the Todd genus of M to be $T_0[M]$. Note that the signature of M is already defined for a torus manifold M because M is an oriented closed manifold, and that it agrees with $T_1[M]$ which follows from the Atiyah-Singer G-signature theorem.

Theorem 10.1. *Let* M *be a torus manifold of dimension* $2n$. *Then*

$$T_y[M] = T_y[\Delta(M)] = \sum_{m=0}^{n} e_{n-m}(\Delta(M))(-1-y)^m.$$

(*See Section 3 for* $e_q(\Delta(M))$.) *In particular, the Todd genus* $T_0[M]$ *of* M *equals* $\deg(\Delta)$.

Proof. Look at the expansion of the right-hand side of (10.1) with respect to y. It follows from (9.2) and Lemma 9.7 that all coefficients of powers of y in (10.1) are elements of $R(T)$. Take a generic vector $v \in H_2(BT)$ and evaluate the right-hand side of (10.1) on v. Then we get the following polynomial in y whose coefficients are Laurent polynomials in z:

$$\sum_{I\in\Sigma^{(n)}} w(M)(I)\frac{\prod_{i\in I}(1+yz^{-\langle u_i^I, v\rangle})}{\prod_{i\in I}(1-z^{-\langle u_i^I, v\rangle})} \tag{10.2}$$

It is easily seen that (10.2) approaches to a polynomial in y with constant coefficents if z tends either to 0 or to ∞. This means that (10.2) itself is a polynomial with constant coefficients. Since v is generic, this implies that (10.1), that is $T_y[M]$, is actually a polynomial with constant coefficients equal to (10.2). Then, by letting z tend to 0, we obtain

$$T_y[M] = \sum_{I\in\Sigma^{(n)}} w(M)(I)(-y)^{\mu(I)},$$

where $\mu(I) = \sharp\{i \in I \mid \langle u_i^I, v\rangle > 0\}$. This $\mu(I)$ agrees with the $\mu(I)$ in Section 3 because $\{u_i^I \mid i \in I\}$ is the dual basis of $\{v_i \mid i \in I\}$. Hence $T_y[M] = T_y[\Delta(M)]$, proving the former identity in the theorem. The latter follows from Corollary 3.3.

As noted in the definition of $T_y[\Delta]$ in Section 3, $T_0[\Delta(M)] = \deg(\Delta(M))$. Since $T_0[M] = T_0[\Delta(M)]$, the last statement in the theorem follows. □

Corollary 10.2. *The signature* $\mathrm{Sign}(M)$ *of a torus manifold* M *is given by*

$$\mathrm{Sign}(M) = \sum_{m=0}^{n} (-2)^m e_{n-m}(\Delta(M)).$$

If $T[M] = 1$ *and* $w(M)(I) = 1$ *for all* $I \in \Sigma(M)^{(n)}$*, then* $e_q(\Delta(M))$ *agrees with the number of cones of dimension* q *in* $\Delta(M)$*.*

Proof. Since $\mathrm{Sign}(M)$ equals $T_1[M]$, the former statement follows from Theorem 10.1. The latter statement is noted in the definition of $e_q(\Delta)$ in Section 3. □

Remark. If M is a compact non-singular toric variety, then $T[M] = 1$ and $w(M)(I) = 1$ for all $I \in \Sigma(M)^{(n)}$, and the formula above is already known in that case ([25, Theorem 3.12(3)]).

11. Equivariant index of a torus manifold

If M is a unitary torus manifold, then the map π collapsing M to a point induces, in equivariant K-theory, an equivariant Gysin homomorphism

$$\pi_!\colon K_T(M) \to K_T(pt) = R(T).$$

If E is a complex T-vector bundle over M, then $\pi_!(E)$ equals the index of a Dirac operator twisted by E. It is sometimes called the equivariant Riemann-Roch number. The Todd genus of M is equal to $\pi_!(1)$.

Let L be a complex T-line bundle over a unitary torus manifold M. Since $\pi_!(L)$ is an element of $R(T)$, one can express

$$\pi_!(L) = \sum_{u\in H^2(BT)} m_L(u)t^u \tag{11.1}$$

with integers $m_L(u)$ which are zero for all but finitely many elements u. In this section we describe the multiplicity $m_L(u)$ of t^u in terms of the (shifted) moment map associated with L when M is a torus manifold. For that, we need to define $\pi_!(L)$ when M is a torus manifold. This is done as follows. When M is a unitary torus manifold, the localization formula applied to $\pi_!(L)$ tells us that

$$\pi_!(L) = \sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)t^{c_1^T(L)|_I}}{\prod_{i\in I}(1-t^{-u_i^I})} \tag{11.2}$$

where $c_1^T(L) \in H_T^2(M)$ denotes the equivariant first Chern class of L. (Note that $t^{c_1^T(L)|_I}$ is nothing but the complex one-dimensional T-representation obtained by restricting L to a point in M_I.) The right-hand side of (11.2) is an element of $R(T)$ by Lemma 9.7 whenever M is a torus manifold although $\pi_!$ may not be defined. Thus we define $\pi_!(L)$ as the right-hand side of (11.2) when M is a torus manifold, and then define $m_L(u)$ as before using (11.1).

In the following, we will make the following assumption on a torus manifold M, which is satisfied for compact non-singular toric varieties with restricted T-actions: *all isotropy subgroups of M are subtori of T and each connected component fixed pointwise by a subtorus contains at least one T-fixed point.* Then the union $\cup_{i=1}^d M_i$ is the set of points with nontrivial isotropy subgroups, and it follows from the slice theorem that the orbit space M/T is a compact connected smooth manifold of dimension n with $\cup_{i=1}^d M_i/T$ as boundary (after corners rounded).

We make a further remark on orientations. The orbit space M/T is orientable (see [23], Lemma 6.7) and we orient it in such a way that the orientation on T followed by that of M/T agrees with that of M times $(-1)^{n(n-1)/2}$. This determines a fundamental class in $H_n(M/T, \partial(M/T))$ and hence in $H_{n-1}(\partial(M/T))$, denoted by $[\partial(M/T)]$, through the boundary homomorphism from $H_n(M/T, \partial(M/T))$ to $H_{n-1}(\partial(M/T))$.

Since $H_T^2(M)$ is additively generated by ξ_i's ($i = 1, \dots, d$) modulo $H^*(BT)$-torsions, $c_1^T(L) = \sum_i c_i\xi_i$ modulo $H^*(BT)$-torsions with some integers c_i's. Associated with L, there is defined the moment map $\Phi_L\colon M \to H^2(BT;\mathbb{R}) = L(T)^*$. It maps M_i into an affine hyperplane $\{u \in H^2(BT;\mathbb{R}) \mid \langle u, v_i\rangle = c_i\}$ for each i (see [23], Lemma 6.5). We slightly shift Φ_L so that the shifted map Φ'_L maps M_i into

$$\mathcal{F}'_L(\{i\}) := \{u \in H^2(BT;\mathbb{R}) \mid \langle u, v_i\rangle = c_i + \frac{1}{2}\}$$

for each i. In fact, Φ'_L is defined as follows. Let K be a complex T-line bundle over M with $c_1^T(K) = -\sum_{i=1}^d \xi_i$. Such K exists ([16]). When M is a compact non-singular toric variety, K is the canonical line bundle of M. Using the moment map $\Phi_K\colon M \to H^2(BT;\mathbb{R})$ associated with K, we define

$$\Phi'_L := \Phi_L - \frac{1}{2}\Phi_K.$$

The moment maps Φ_L and Φ_K are equivariant, the T-action on the target $H^2(BT;\mathbb{R})$ being trivial; so Φ'_L induces a map

$$\bar{\Phi}'_L\colon M/T \to H^2(BT;\mathbb{R}).$$

The shifted affine hyperplanes $\mathcal{F}'_L(\{i\})$'s miss the lattice $H^2(BT)$. Since $\partial(M/T) = \cup_i(M_i/T)$ and $\bar{\Phi}'_L$ maps M_i/T to $\mathcal{F}'_L(\{i\})$ for each i, $\bar{\Phi}'_L$ induces a homomorphism

$$(\bar{\Phi}'_L)_*\colon H_{n-1}(\partial(M/T)) \to H_{n-1}(H^2(BT;\mathbb{R})\backslash\{u\})$$

for each lattice point $u \in H^2(BT)$. We define

$$d'_L(u) := \text{ the mapping degree of } (\bar{\Phi}'_L)_*$$

where the orientation on $H^2(BT;\mathbb{R})$ is determined by that on T. Our main theorem in this section is the following.

Theorem 11.1. *Let M be a torus manifold. Then $m_L = d'_L$ on $H^2(BT)$.*

Remark. This theorem was first established by Karshon-Tolman [18] when M is a compact non-singular toric variety, and then extended to a unitary torus manifold by the second named author [23], while Grossberg-Karshon [10] extends the result of [18] to Spin^c manifolds with torus actions. The family of torus manifolds contains these manifolds.

Let $S(M)$ be the realization of the first barycentric subdivision of $\Sigma(M)$ and let $S(M)_i$ be the union of simplicies in $S(M)$ which contain the vertex $\{i\}$ as in Section 6. Since $S(M)_I = \cap_{i\in I} S(M)_i$ is contractible for any non-empty set $I \in \Sigma(M)$ and $\partial(M/T) = \cup_{i=1}^d (M_i/T)$, it follows from Lemma 6.2 that there is a continuous map

$$\rho_M\colon \partial(M/T) \to S(M)$$

sending $\cap_{i\in I}(M_i/T)$ to $S(M)_I$ for each $I \in \Sigma(M)$, and that such a map is unique up to homotopy preserving the stratifications, where the stratifications on $\partial(M/T)$ and $S(M)$ mean subspaces $\cap_{i\in I}\partial(M_i/T)$ and $S(M)_I$ indexed by elements I's in $\Sigma(M)$.

If the orientation on T or M is reversed, then $[\partial(M/T)]$ and $[\Delta(M)]$ will be multiplied by -1 simultaneously; so the following lemma makes sense.

Lemma 11.2. $\rho_{M*}([\partial(M/T)]) = [\Delta(M)]$.

Proof. We prove the lemma by induction on the dimension $n = \dim(M/T)$. When $n = 1$, M is S^2 with a nontrivial smooth S^1-action. In this case, it is not difficult to check the lemma, which we leave to the reader.

Since a characteristic submanifold of M_i is a connected component of $M_i \cap M_j$ for some j and such j is uniquely determined by the characteristic submanifold of M_i, there is a natural map $\pi_i\colon \Sigma(M_i) \to \Sigma(M)_i$. This map is an isomorphism if $M_i \cap M_j$ is connected for any j, but otherwise it is only surjective. As we did in Lemma 6.1, we identify the realization of $\Sigma(M)_i$ with $\partial(S(M)_i)$. One sees that

$$\pi_{i*}([\Delta(M_i)]) = \sum_{i\in I\in\Sigma(M)^{(n)}} w(M)(I)\langle I\backslash\{i\}\rangle \in H_{n-2}(\partial(S(M)_i)) = H_{n-2}(\Sigma(M)_i). \tag{11.3}$$

Since M_i is itself a torus manifold, the spaces $\partial(M_i/T)$ and $S(M_i)$ have stratifications like for M, and hence we have a map $\rho_{M_i}\colon \partial(M_i/T) \to S(M_i)$ preserving the stratifications. By the induction assumption

$$\rho_{M_i*}([\partial(M_i/T)]) = [\Delta(M_i)] \in H_{n-1}(S(M_i)) = H_{n-1}(\Sigma(M_i)). \tag{11.4}$$

On the other hand, $\partial(S(M)_i)$ has a stratification induced from $S(M)$ and each stratum is contractible. Since ρ_M restricted to $\partial(M_i/T)$ is a map from $\partial(M_i/T)$ to $\partial(S(M)_i)$ preserving the stratifications and so is $\pi_i \circ \rho_{M_i}$ as well, they are homotopic preserving the stratifications by Lemma 6.2. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccc}
H_{n-1}(\partial(M/T)) & \xrightarrow{\text{injective}} & \bigoplus_i H_{n-1}(M_i/T, \partial(M_i/T)) & \xrightarrow{\cong} & \bigoplus_i H_{n-2}(\partial(M_i/T)) \\
{\scriptstyle \rho_{M*}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \oplus\pi_{i*}\rho_{M_i*}} \\
H_{n-1}(S(M)) & \xrightarrow{\text{injective}} & \bigoplus_i H_{n-1}(S(M)_i, \partial(S(M)_i)) & \xrightarrow{\cong} & \bigoplus_i H_{n-2}(\partial(S(M)_i))
\end{array}$$

where the left horizontal maps are restrictions. Tracing the upper horizontal sequence from the left to the right, $[\partial(M/T)] \in H_{n-1}(\partial(M/T))$ maps to $\bigoplus_i [\partial(M_i/T)]$, and down to $\sum_{i\in I\in\Sigma(M)^{(n)}} w(M)(I)\langle I\backslash\{i\}\rangle \in \bigoplus_i H_{n-2}(\partial(S(M)_i))$ by (11.3) and (11.4), while $[\Delta(M)] \in H_{n-1}(S(M))$ maps through the lower horizontal sequence to the same element as observed in Lemma 6.1. Since the horizontal sequences above are injective, the lemma follows. □

Proof of Theorem 11.1. By Lemma 6.2 we have a map $S(M) \to H^2(BT;\mathbb{R})$ associated with the multi-polytope $\mathcal{P}'_L := (\Delta(M), \mathcal{F}'_L)$. We denote the map by Ψ'_L. The composition $\Psi'_L \circ \rho_M$ is a map from $\partial(M/T)$ to $H^2(BT;\mathbb{R})$ sending $\cap_{i\in I}(M_i/T)$ to $\cap_{i\in I}\mathcal{F}'_L(\{i\})$ for any $I \in \Sigma(M)$, and so is $\bar{\Phi}'_L$ as well. Therefore, $\Psi'_L \circ \rho_M$ and $\bar{\Phi}'_L$ are homotopic preserving the stratifications by Lemma 6.2. It follows from Lemma 11.2 that

$$\begin{aligned} d'_L(u) &= \text{the mapping degree of } (\bar{\Phi}'_L)_* : H_{n-1}(\partial(M/T)) \to H_{n-1}(H^2(BT;\mathbb{R})\backslash\{u\}) \\ &= \text{the mapping degree of } (\Psi'_L \circ \rho_M)_* : H_{n-1}(\partial(M/T)) \to H_{n-1}(H^2(BT;\mathbb{R})\backslash\{u\}) \\ &= \text{the mapping degree of } (\Psi'_L)_* : H_{n-1}(S(M)) \to H_{n-1}(H^2(BT;\mathbb{R})\backslash\{u\}) \\ &= \mathrm{WN}_{\mathcal{P}'_L}(u) = \mathrm{DH}_{\mathcal{P}'_L}(u) = \mathrm{DH}_{(\mathcal{P}_L)_+}(u). \end{aligned}$$

This together with Corollary 7.4 and the definition of m_L (i.e., (11.1) and (11.2)) proves the theorem. □

12. Torus orbifolds

We first recall basic definitions concerning orbifolds. We refer to [27], [20] or [8] for details. The reference [22] will be also useful; it deals with torus actions on symplectic orbifolds. If M is an orbifold of dimension n, then there is a family $\{(U_\alpha, V_\alpha, H_\alpha, p_\alpha)\}$ of orbifold charts, where $\{U_\alpha\}$ is an open covering of M, V_α is an n-dimensional manifold, H_α is a finite group acting on V_α and $p_\alpha : V_\alpha \to U_\alpha$ is a map which induces a homeomorphism from V_α/H_α onto U_α. If U_α and U_β intersect each other, then the charts $(U_\alpha, V_\alpha, H_\alpha, p_\alpha)$ and $(U_\beta, V_\beta, H_\beta, p_\beta)$ satisfy suitable compatibility conditions. Such a family $\{(U_\alpha, V_\alpha, H_\alpha, p_\alpha)\}$ is called an orbifold atlas. For any point x in M, there exists a special type of orbifold chart (U_x, V_x, H_x, p_x) with the property that $p_x^{-1}(x)$ is a single point $\tilde{x} \in V_x$. The isomorphism class of the group H_x is uniquely determined by x and is called the isotropy group of x. The order of H_x, denoted by d_x, is called the multiplicity of the point x. Such an orbifold chart will be called a special orbifold chart. When M is connected, the minimum of the multiplicities is called the multiplicity of the orbifold M and is denoted by $d(M)$. The set $\{x \in M \mid d_x = d(M)\}$ is open and dense in M. It is a manifold. This set is called the principal stratum of the orbifold M. We have $d(M) = 1$ if and only if the actions of all the isotropy groups are effective.

A map $f : M \to M'$ from an orbifold M into another orbifold M' is called smooth if, near every point x in M, there is a smooth map $f_\alpha : V_\alpha \to V'_\alpha$ for suitable orbifold charts $(U_\alpha, V_\alpha, H_\alpha, p_\alpha)$ for M around x and $(U'_\alpha, V'_\alpha, H'_\alpha, p'_\alpha)$ for M' around $f(x)$ satisfying the commutativity relation $p'_\alpha \circ f_\alpha = f \circ p_\alpha$. A subset M of an orbifold M' is called a suborbifold if, for each orbifold chart $(U'_\alpha, V'_\alpha, H'_\alpha, p'_\alpha)$ of M', $V_\alpha = {p'_\alpha}^{-1}(M \cap U'_\alpha)$ is an H'_α-invariant submanifold of V'_α. If this is the case, M becomes an orbifold with orbifold charts $(U_\alpha, V_\alpha, H'_\alpha, p'_\alpha)$ where $U_\alpha = M \cap U'_\alpha$, and the inclusion $M \to M'$ becomes a smooth map. It may happen that $d(M) > d(M')$ (M and M' are assumed connected). The integer $d(M|M') = d(M)/d(M')$ will be called the relative multiplicity of M (with respect to M').

Orbifold vector bundles are also defined. Typical examples are the tangent bundle of an orbifold and the normal bundle of a suborbifold. An orbifold is orientable if its tangent bundle is orientable. If $E \to M$ is an orbifold vector bundle over a connected orbifold, then the relative multiplicity of the orbifold vector bundle E is defined to be $d(M|E)$, the relatvie multiplicity of the zero cross-section M regarded as a suborbifold of E. If

M is a suborbifold of M' and ν is the normal bundle of M in M', then $d(M|\nu)$ equals $d(M|M')$.

Let G be a Lie group. An action of G on an orbifold M is a smooth map $\psi : G\times M \to M$ satisfying the usual rule of group action. Suppose that G is connected. If $x \in M$ is a fixed point of the action, and (U_x, V_x, H_x, p_x) is a special orbifold chart around x such that U_x is invariant under the action of G, then there is a finite covering group $\tilde{G}_x$ of G and an action of $\tilde{G}_x$ on V_x which covers the action of G on U_x. If G is compact, the fixed point set of the action is a suborbifold.

Now let M be an oriented, closed orbifold of dimension $2n$ with an effective action of an n-dimensional torus T. A connected component of the fixed point set by a circle subgroup is a suborbifold. A suborbifold of this type which has codimension two and contains at least one fixed point of the T-action will be called a *characteristic suborbifold.* Let M_i be a characteristic suborbifold and $x \in M_i$. We take, as we may, a special orbifold chart (U_x, V_x, H_x, p_x) around x such that V_x is an open disk in $\mathbb{R}^{2n}$ and the action of H_x on V_x is linear. We denote by the same symbol V_x the tangent space to V_x at the point $\tilde{x} = p_x^{-1}(x)$. Then the vector space V_x decomposes into a direct sum $V_{ix} \oplus V_{ix}^{\perp}$ where $V_{ix}^{\perp}$ is tangent to $p_x^{-1}(U_x \cap M_i)$, and the vector space V_{ix} represents the fiber direction of the normal bundle of M_i in M. The isotropy group H_x acts on V_{ix}.

Lemma 12.1. *Let M be an oriented closed orbifold as above and M_i a characteristic suborbifold. Let S_i denote the circle subgroup which fixes the points of M_i. Then there exists a finite covering group $\tilde{S}_i$ of S_i and a lifting of the action of S_i to the action of $\tilde{S}_i$ on V_x for any point $x \in M_i$. The lifted action of $\tilde{S}_i$ preserves V_{ix}.*

Proof. To $x \in M_i$ we correspond the degree of the minimal finite covering $\tilde{S}_{ix}$ of S_i such that there is a lifting of the action to $\tilde{S}_{ix}$. The lifted action necessarily preserves V_{ix}. It is not difficult to see that the correspondence is locally constant. Since M_i is connected the correspondence must be constant. □

Hereafter we denote by $\rho_i : \tilde{S}_i \to S_i$ the minimal finite covering of S_i with the above property. $\tilde{S}_i$ acts effectively on V_x.

An oriented, closed orbifold M of dimension $2n$ with an effective action of a torus T of dimension n with non-empty fixed point set M^T equipped with a preferred orientation of the normal bundle of each characteristic suborbifold will be called a *torus orbifold* if, for each M_i and at each point $x \in M_i$, the action of H_x preserves the orientation of each V_{ix}. Note that choosing an orientation of a characteristic submanifold is equivalent to choosing an orientation of its normal bundle. Thus a torus manifold is a torus orbifold in the above sense. Another example is a unitary torus orbifold. A unitary torus orbifold is a torus orbifold such that V_α is a unitary manifold, the action of H_α preserves the unitary structure of V_α for each orbifold chart $(U_\alpha, V_\alpha, H_\alpha, p_\alpha)$ and the action of T on M also preserves the unitary structure of V_α's.

Let M be a torus orbifold. The preferred orientation of the normal bundle ν_i of M_i makes it a complex orbifold line bundle. Then there is a unique isomorphism $\varphi_i : S^1 \to \tilde{S}_i$ such that $\varphi(z)$ acts by the complex multiplication of z on each V_{ix}. We identify $\tilde{S}_i$ with S^1 via φ_i. The homomorphism $\rho_i : S^1 = \tilde{S}_i \to T$ defines an element $v_i \in \mathrm{Hom}(S^1, T) = H_2(BT; \mathbb{Z})$. We are now ready to define the multi-fan $\Delta(M) = (\Sigma(M), C(M), w(M)^{\pm})$ associated with a torus orbifold M in an entirely similar way to the case of torus manifolds. Specifically

$$\Sigma(M) = \{I \mid (\cap_{i\in I} M_i)^T \neq \emptyset\},$$

and $C(M)(I)$ is the cone in $H_2(BT;\mathbb{R})$ with apex at 0 and spanned by $\{v_i \mid i \in I\}$. Furthermore $w(M)^{\pm}(I) = \#\{x \in M_I \mid \epsilon_x = \pm 1\}$ for $I \in \Sigma(M)^{(n)}$, where ϵ_x is defined to be the ratio of two orientations at x, one which is given by the orientation of M and the other by that of the oriented vector space $V_x = \oplus_{i\in I} V_{ix}$.

We set $\tilde{T}_I = \prod_{i\in I} \tilde{S}_i$ for $I \in \Sigma(M)^{(k)}$ and $\rho_I = \prod_{i\in I} \rho_i : \tilde{T}_I \to T$. The image of ρ_I is denoted by T_I. $\rho_I : \tilde{T}_I \to T_I$ is a finite covering. T_I fixes the points of $M_I = \bigcap_{i\in I} M_i$. If $I \in \Sigma(M)^{(n)}$, then $T_I = T$. Let x be a fixed point of the action of T on M. Then there is a unique $I \in \Sigma(M)^{(n)}$ such that x belongs to M_I. The inclusion $S^1 = \tilde{S}_i \to \tilde{T}_I$ defines an element $\tilde{v}_i \in \mathrm{Hom}(S^1, \tilde{T}_I) = H_2(B\tilde{T}_I;\mathbb{Z})$, and we have $\rho_{I*}(\tilde{v}_i) = v_i$. V_x and V_{ix}, $i \in I$, are complex $\tilde{T}_I$-modules, and the decomposition $V_x = \oplus_{i\in I} V_{ix}$ is compatible with the action of $\tilde{T}_I$. The effectiveness of the T-action on M implies that $\tilde{T}_I$ effectively acts on V_x; equivalently, it implies that $\{\tilde{v}_i \mid i \in I\}$ is a basis of $H_2(B\tilde{T}_I;\mathbb{Z})$. Since $\rho_{I*} : H_2(B\tilde{T}_I;\mathbb{Z}) \to H_2(BT;\mathbb{Z})$ is injective, the v_i, $i \in I$, are linearly independent in $H_2(BT;\mathbb{R})$.

Lemma 12.2. *$\Delta(M)$ is a complete multi-fan.*

Proof. The argument is almost similar to the case of torus manifolds. One has only to observe that the characteristic suborbifolds and their intersections are torus orbifolds and a 2-dimensional torus orbifold is topologically a 2-sphere acted on by a circle group with exactly two fixed points. □

Lemma 12.3. *Suppose $d(M) = 1$. Let $I \in \Sigma(M)^{(k)}$, and let x be a point in the principal stratum (as an orbifold) of M_I. Then the isotropy group H_x of x is isomorphic to the kernel of $\rho_I : \tilde{T}_I \to T$.*

Proof. Let (U_x, V_x, H_x, p_x) be an orbifold chart around x. We may regard V_x as an n dimensional $\tilde{T}_I$-module as before. As such V_x is decomposed as a direct sum of $\tilde{T}_I$-modules

$$V_x = (\oplus_{i\in I} V_{ix}) \oplus V'$$

where V' is projected into M_I by p_x. $\tilde{T}_I = \prod_{i\in I} \tilde{S}_i$ can be regarded as embedded in the general linear group of $\oplus_{i\in I} V_{ix}$. Since H_x acts on each V_{ix} preserving its orientation, there is a homomorphism $H_x \to \tilde{T}_I$. The action of H_x on V' is trivial. Moreover the action of H_x on V_x is effective because $d(M) = 1$. It follows that the homomorphism above embeds H_x into $\tilde{T}_I$. Since the kernel of ρ_I is equal to the intersection of $\tilde{T}_I$ with the image of H_x, it is isomorphic to H_x. □

It is known that a closed oriented orbifold M of dimension n has the fundamental class $[M] \in H_n(M;\mathbb{Z})$, and that the Poincaré duality holds, i.e., the operation $\vartheta = [M]\cap : H^q(M;\mathbb{Q}) \to H_{n-q}(M;\mathbb{Q})$ is an isomorphism. If $f : M \to M'$ is a smooth map from an oriented close orbifold M to another such M', then the Gysin homomorphism $f_! : H^q(M;\mathbb{Q}) \to H^{q+n-n'}(M';\mathbb{Q})$ is defined to be the compostion $\vartheta^{-1} \circ f_* \circ \vartheta$, where n' is the dimension of M'. If a compact Lie group G acts on M and M', and f is equivariant, then the equivariant Gysin homomorphism $f_! : H^q_G(M;\mathbb{Q}) \to H^{q+n-n'}_G(M';\mathbb{Q})$ is also defined.

Henceforth M will be a torus orbifold. For each $i \in \Sigma(M)^{(1)}$, we set

$$\xi_i = (f_i)_!(1) \in H^2_T(M;\mathbb{Q}),$$

where $f_i : M_i \to M$ is the inclusion.

Lemma 12.4. *Let $c_1^T(\nu_i)$ be the equivariant first Chern class of the normal bundle ν_i. Then we have*

$$c_1^T(\nu_i) = f_i^*(\xi_i).$$

Proof. We may assume that $d(M) = 1$. Take an equivariant Thom form ϕ for the equivariant orbifold bundle ν_i (we refer to [3] for Thom form and Chern form). Let x be a point in the principal stratum of M_i, and (U_x, V_x, H_x, p_x) an orbifold chart around x. The restriction of ϕ to V_x is invariant under the action of H_x and its support is contained in a tubular neighborhood W_i of $V_i = p_x^{-1}(U_i)$, where $U_i = U_x \cap M_i$. Moreover, with respect to the fibering $\tilde{\pi}_i : W_i \to V_i$, we have $|H_x|^{-1}(\tilde{\pi}_i)_*(\phi) = 1$, where $(\tilde{\pi}_i)_*$ is the integration along the fiber of $\tilde{\pi}_i$. Note that the fiber is V_{ix}, and that the action of H_x preserves the orientation of V_{ix}. The equivariant Chern class $c_1^T(\nu_i)$ is the restriction to M_i of the cohomology class $[\phi]$ of ϕ. Here $[\phi]$ is considered as a relative class in $H_T^2(W, W \setminus M_i; \mathbb{R})$ where W is a tubular neighborhood of M_i.

On the other hand, ξ_i is the restriction of a cohomology class $\psi \in H_T^2(W, W \setminus M_i; \mathbb{R})$ such that

$$\pi_*(\psi) = 1 \in H_T^0(W; \mathbb{R}) = H_T^0(M_i; \mathbb{R}),$$

where $\pi : W \to M_i$ denotes the projection of the fibration. Note that the fiber of π is $U_{ix} = V_{ix}/H_x$, where H_x acts effectively on V_{ix}. We have

$$\pi_*([\phi]) = |H_x|^{-1}(\tilde{\pi}_i)_*([\phi]) = 1 = \pi_*(\psi).$$

But π_* is an isomorphism (Thom isomorphism). Hence we have $[\phi] = \psi$, and consequently

$$c_1^T(\nu_i) = [\phi]|M_i = \psi|M_i = f_i^*(\xi_i).$$

□

We noticed that, for $I \in \Sigma(M)_{(n)}$, $\{v_i \mid i \in I\}$ was a basis of $H_2(BT; \mathbb{R})$. Let $\{u_i^I\}$ be the dual basis in $H^2(BT; \mathbb{R})$. This can be interpreted in the following way. Let $\{\tilde{u}_i \mid i \in I\}$ be the basis of $H^2(B\tilde{T}_I; \mathbb{Z})$ dual to $\{\tilde{v}_i \mid i \in I\}$. We have $\rho_I^*(u_i^I) = \tilde{u}_i$, since $\rho_{I*}(\tilde{v}_i) = v_i$. We identify $H^2(B\tilde{T}_I; \mathbb{R})$ with $H^2(BT; \mathbb{R})$ by the isomorphism ρ_I^*. Then $H^2(B\tilde{T}_I; \mathbb{Z})$ can be considered as embedded in $H^2(BT; \mathbb{R})$. With this convention we have $u_i^I = \tilde{u}_i$.

Let $x \in M^T$ be a fixed point of the T-action. In the sequel we identify $H_T^2(x; \mathbb{R})$ with $H^2(BT; \mathbb{R})$.

Lemma 12.5. *Let $I \in \Sigma(M)^{(n)}$ and $x \in M_I$. Then $\xi_i|x = u_i^I \in H^2(BT; \mathbb{R})$ for $i \in I$. If $j \notin I$, then $\xi_j|x = 0$.*

Proof. By Lemma 12.4 we have

$$\xi_i|x = c_1^T(\nu_i|x).$$

But $\nu_i|x$ viewed as $\tilde{T}_I$-module is V_{ix}. It follows that $c_1^{\tilde{T}_I}(\nu_i|x) = \tilde{u}_i$. Hence

$$c_1^T(\nu_i|x) = u_i^I.$$

If $j \notin I$, then $x \notin M_j$. Therefore $\xi_j|x = 0$. □

If we consider $u_i^I = \tilde{u}_i$ as an element of $\mathrm{Hom}(\tilde{T}_I, S^1) = H^2(B\tilde{T}_I; \mathbb{Z})$, then Lemmas 12.5 and 12.6 imply that u_i^I is nothing but the $\tilde{T}_I$-module V_{ix}. The following Lemma describes the algebra structure of $H_T^*(M; \mathbb{R})$ over $H^*(BT; \mathbb{R})$ modulo $H^*(BT; \mathbb{R})$-torsion as in the case of torus manifolds (Lemma 9.3).

Lemma 12.6. *The following equality holds for any $u \in H^2(BT;\mathbb{R})$:*

$$\pi^*(u) = \sum_{i\in\Sigma(M)^{(1)}} \langle u, v_i\rangle \xi_i \quad \textit{modulo } H^*(BT;\mathbb{R})\textit{-torsion.}$$

Proof. Let $x \in M_I \subset M^T$ be a fixed point of the T-action. We restrict both sides of the equality in Lemma 12.6 to x. On the left hand side we get u. On the right hand side the result is

$$\sum_{i\in I} \langle u, v_i\rangle u_i^I$$

by virtue of Lemma 12.5. But this is equal to u by the definition of the u_i^I. Thus both sides coincide after the restriction to each $x \in M^T$. Since the restriction homomorphism $\pi^* : H_T^*(M;\mathbb{R}) \to H_T^*(M^T;\mathbb{R})$ is injective modulo $H^*(BT;\mathbb{R})$-torsion, the equality is confirmed. □

Remark. The equality in Lemma 12.6 characterizes the vectors v_i in terms of the ξ_i as in Lemma 9.3.

We set $N = H_2(BT;\mathbb{Z})$ and define N_I for $I \in \Sigma(M)^{(n)}$ to be the lattice generated by the $v_i,\ i \in I$.

Lemma 12.7. *Assume that $d(M) = 1$. Let $x \in M_I$ with $I \in \Sigma(M)^{(n)}$. Then H_x is isomorphic to* $\operatorname{Ker}\rho_I$. *Moreover* $\operatorname{Ker}\rho_I$ *is isomorphic to N/N_I.*

Proof. We have already shown that H_x is isomorphic to the kernel of ρ_I in Lemma 12.3. For the second part it suffices to note that N and N_I can be identified with the fundamental group of T and $\tilde{T}_I$. Therefore the kernel of ρ_I is isomorphic to N/N_I. □

Remark. Hereafter we identify H_x and N/N_I with $\operatorname{Ker}\rho_I \subset \tilde{T}_I$ through the isomorphisms given in Lemma 12.7. We put $\chi_I(u,v) = \exp(2\pi\sqrt{-1}\langle u,v\rangle)$ for $u \in H^2(B\tilde{T}_I;\mathbb{Z})$ and $v \in H_2(BT;\mathbb{R})$. If u is fixed, then the value $\chi_I(u,v)$ depends only on the equivalence class of v modulo N_I. Hence, if we identify $\tilde{S}_i$ with S^1 via φ_i as before and $\tilde{T}_I$ with $\prod_{i\in I} S^1$ via $\prod_{i\in I}\varphi_i$, then the map $\exp : H_2(BT;\mathbb{R}) \to \tilde{T}_I$ defined by $\exp(v) = \prod_{i\in I}\exp(2\pi\sqrt{-1}\langle u_i^I, v\rangle)$ is a universal covering map and its kernel is N_I. It induces an isomorphism from $H_x = N/N_I$ onto $\operatorname{Ker}\rho_I$. We shall write $\chi_I(u,g)$ instead of $\chi_I(u,v)$ for $g = \exp(v) \in \tilde{T}_I$ as in Section 7. Let V be a one dimensional $\tilde{T}_I$-module. It defines an element $u \in \operatorname{Hom}(\tilde{T}_I, S^1) = H^2(B\tilde{T}_I;\mathbb{Z})$. Then the action of $g \in \tilde{T}_I$ on V is given by the complex multiplication by $\chi_I(u,g)$.

Suppose that M is a unitary torus orbifold such that $d(M) = 1$. Let L be a T-invariant complex line bundle over M. By using the hermitian connection of M and a hermitian connection of L, a Dirac operator twisted by L is defined as in the case of torus manifolds. Its index is a T-module. It is called the equivariant Riemann-Roch number with coefficient in L, and is denoted by $RR^T(M,L) \in R(T)$. It can be expressed by the fixed point formula due to Vergne [29]; cf. also [8]. The formula is particularly simple when all the fixed points are isolated. It is convenient to write down the image of $RR^T(M,L)$ by $\operatorname{ch} : R(T) \to H^{**}(BT;\mathbb{R})$; the result is

Lemma 12.8. *Let $\xi = c_1^T(L)$ be the equivariant Chern class of L. Then*

$$\operatorname{ch}(RR^T(M,L)) = \sum_{x\in M^T} \frac{\epsilon_x e^{\xi|x}}{|H_x|} \sum_{g\in H_x} \frac{\chi_{I_x}(\xi|x, g)}{\prod_{i\in I_x}(1-\chi_{I_x}(u_i^{I_x}, g)^{-1} e^{-u_i^{I_x}})},$$

where $I_x \in \Sigma(M)^{(n)}$ *is such that* $x \in M_{I_x}$.

It can be shown that, if x and y both lie in the same M_I, then $\xi|x = \xi|y$ for $\xi = c_1^T(L)$. The proof is same as in the case of torus manifolds as was given in [23]. We shall write $u_I = c_1^T(L)|x$ for $x \in M_I$. Taking Remark below Lemma 12.7 in account, we get

Proposition 12.9.

$$\mathrm{ch}(RR^T(M,L)) = \sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)e^{u_I}}{|N/N_I|} \sum_{g\in N/N_I} \frac{\chi_I(u_I,g)}{\prod_{i\in I}(1-\chi_I(u_i^I,g)^{-1}e^{-u_i^I})} .$$

Since $\mathrm{ch}: R(T) \to H^{**}(BT;\mathbb{R})$ is injective, the formula in Proposition 12.9 characterizes $RR^T(M,L)$. Using the notation in Section 7, we obtain

Corollary 12.10.

$$RR^T(M,L) = \sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)t^{u_I}}{|N/N_I|} \sum_{g\in N/N_I} \frac{\chi_I(u_I,g)}{\prod_{i\in I}(1-\chi_I(u_i^I,g)^{-1}t^{-u_i^I})} .$$

When $u_I = c_1^T(L)|x$, $x \in M_I$, lies in $N^* = H^2(BT;\mathbb{Z})$, then $\chi_I(u_I,g) = 1$ for all $g \in N/N_I$. Therefore, if $u_I \in N$ for all $I \in \Sigma(M)^{(n)}$, then

$$RR^T(M,L) = \sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)t^{u_I}}{|N/N_I|} \sum_{g\in N/N_I} \frac{1}{\prod_{i\in I}(1-\chi_I(u_i^I,g)^{-1}t^{-u_i^I})} .$$

By observing that $g \mapsto \chi_I(u,g)$ is a character of N/N_I for any $u \in H^2(B\tilde{T}_I,\mathbb{Z}) = \mathrm{Hom}(\tilde{T}_I,S^1)$, the formula above can be rewritten in the following form:

$$RR^T(M,L) = \sum_{I\in\Sigma(M)^{(n)}} \frac{w(M)(I)t^{u_I}}{|N/N_I|} \sum_{g\in N/N_I} \frac{1}{\prod_{i\in I}(1-\chi_I(u_i^I,g)t^{-u_i^I})} . \tag{12.1}$$

The right hand side of this formula (12.1) appeared in Corollary 7.4. There, it was related to a lattice multi-polytope $\mathcal{P}$ and the Duistermaat-Heckman function $DH_{\mathcal{P}_+}$. Suppose that $c_1^T(L)$ is of the form $c_1^T(L) = \sum_{i\in\Sigma(M)^{(1)}} c_i\xi_i$. Then we define $\mathcal{F}(i)$ to be the hyperplane in $H^2(BT;\mathbb{R})$ defined by

$$\mathcal{F}(i) = \{u \in H^2(BT;\mathbb{R}) \mid \langle u, v_i\rangle = c_i\}$$

for each $i \in \Sigma(M)^{(1)}$. The multi-polytope $\mathcal{P} = (\Delta(M),\mathcal{F})$ is the one mentioned in Corollary 7.4. Note that $\mathcal{P}$ is not always a lattice multi-polytope in this case.

Remark. Corollary 7.4 shows that the right hand side of the formula (12.1) depends only on $\Delta(M)$ and $\mathcal{P}$; namely, it does not depend on the choice of generating vectors $v_i \in H^2(BT;\mathbb{R})$ in so far as they lie in $N = H^2(BT;\mathbb{Z})$ and $\{u_i^I \mid i \in I\}$ is interpreted as the dual basis of $\{v_i \mid i \in I\}$.

When M is a torus manifold, the Duistermaat-Heckman function has a geometric meaning coming from the moment map of the line bundle L as was explained in Section 11. There it is also explained the role of the winding number. These notions are generalized to the case of torus orbifolds and similar results hold in this case too. The details can be worked out without much alteration and are left to the reader.

13. Realizing multi-fans by torus orbifolds

In the previous section, we associated a complete simplicial multi-fan of dimension n with a torus orbifold of dimension $2n$. In this section, we consider the converse problem. If a multi-fan Δ is associated with a torus orbifold M, then we say that Δ is *(geometrically) realized* by M, or M *realizes* Δ.

We recall how the multi-fan of M changes when the orientations on M or M_i are reversed. If the orientation of M is unchanged but that of M_i is reversed, then the orientation of the normal bundle of M_i is reversed and, hence, 1-dimensional cone $C(i)$ tunrs into the cone $-C(i)$, and the pair $(w(M)^+(I), w(M)^-(I))$ turns into $(w(M)^-(I), w(M)^+(I))$ for $I \in \Sigma(M)^{(n)}$ containing i while others remain unchanged. If the orientations of M and of all the M_i's are reversed, then all the cones $C(i)$'s remain unchanged but $(w(M)^+(I), w(M)^-(I))$ turns into $(w(M)^-(I), w(M)^+(I))$ for any $I \in \Sigma(M)^{(n)}$ so that $w(M)(I)$ turns into $-w(M)(I)$ for any $I \in \Sigma(M)^{(n)}$. The torus orbifold M with the reversed orientations of M and all the M_i's will be denoted by $-M$.

The underlying space of a torus orbifold of dimension 2 is S^2 with the standard S^1-action. In this case, there are two characteristic submanifolds. They are S^1-fixed points. Taking orientations on S^2 and its characteristic submanifolds into account, we easily have the following theorem.

Theorem 13.1. *A complete simplicial multi-fan $\Delta = (\Sigma, C, w^\pm)$ of dimension 1 is geometrically realized if and only if Σ is isomorphic to the argumented simplicial set obtained from the boundary of a 1-simplex and $\{w^+(I), w^-(I)\} = \{1, 0\}$ as a set for $I \in \Sigma^{(1)}$.*

The analysis of a torus orbifold of dimension 4 is more complicated. In this case, each characteristic suborbifold is homeomorphic to S^2 and has two fixed points. Therefore, if two of the characteristic suborbifolds intersect, then they intersect at one point or two points, and if they intersect at two points, then they do not intersect at any other characteristic suborbifolds. We also note that a T-fixed point is an intersection of two characteristic suborbifolds. These facts imply the "only if" part in the following theorem. We will prove the "if" part later.

Theorem 13.2. *A complete simplicial multi-fan $\Delta = (\Sigma, C, w^\pm)$ of dimension 2 is geometrically realized if and only if the following two conditions are satisfied for each $I \in \Sigma^{(2)}$:*

(1) *$\{w^+(I), w^-(I)\} = \{1, 0\}$ or $\{1, 1\}$,*
(2) *when $\{w^+(I), w^-(I)\} = \{1, 0\}$, there are exactly two elements, say I' and I'', in $\Sigma^{(2)}$ such that $I \cap I'$ and $I \cap I''$ are in $\Sigma^{(1)}$ and $I \cap I' \cap I'' = \emptyset$, and when $\{w^+(I), w^-(I)\} = \{1, 1\}$, there is no element $I' \in \Sigma^{(2)}$ such that $I \cap I' \in \Sigma^{(1)}$.*

In contrast to the low dimensional cases above, we have

Theorem 13.3. *Any complete simplicial multi-fan of dimension ≥ 3 is geometrically realized.*

In the following $\Delta = (\Sigma, C, w^\pm)$ will be a complete simplicial multi-fan of dimension $n \geq 2$ unless otherwise stated. Here is an outline of how to realize Δ by a torus orbifold. We choose and fix a generic (rational) 1-dimensional cone in $N_{\mathbb{R}}$, and decompose Δ using it into a number of what we call *minimal* multi-fans. Minimal multi-fans can essentially be realized by weighted projective spaces. We paste them together by performing equivariant connected sum along characteristic suborbifolds and at T-fixed points to obtain a desired torus orbifold realizing the given Δ.

Equivariant connected sum is performed through two isomorphic orbifold charts. In this way attention should be payed to orbifold structures. So we make a remark on orbifold structures at this point. There are many choices of an orbifold structure on M (e.g. S^2 with the standard S^1-action admits infinitely many orbifold structures), but the associated multi-fan does not depend on the choice of an orbifold structure. In fact, the circle subgroup S_i determined by the vector v_i in the previous section is the one which fixes points in the characteristic suborbifold M_i, so the line generated by v_i is independent of the orbifold structure. Moreover the direction of v_i is determined by the choice of orientations on M and M_i, so the cone spanned by v_i is independent of the orbifold structure. What depends on the orbifold structure is the length of v_i which is equal to the degree of the covering map $\tilde{S}_i \to S_i$. In this way the vectors v_i reflect the orbifold structure related to the torus action. We shall call the vector v_i the edge vector of the 1-dimensional cone $C(i)$.

We shall use two types of equivariant connected sum; one is at T-fixed points and the other is along characteristic suborbifolds. Let us explain the former first. Suppose that torus orbifolds M and M' with $d(M) = d(M')$ have T-fixed points q and q' respectively such that the n-dimensional cones and the edge vectors corresponding to them are the same and the signs ϵ_q and $\epsilon_{q'}$ at q and q' are opposites. Then there are a finite covering $\tilde{T}$ of T, a finite subgroup H of $\tilde{T}$ and orbifold charts (U, V, H, p) and (U', V, H, p') around q and q' respectively such that V is an invariant open disk centered at the origin in a $\tilde{T}$-module. In particular a diffeomorphism (in the sense of orbifold) f from the closure of U onto that of U' is induced. Moreover f sends characteristic suborbifolds that contain q onto characteristic suborbifolds that contain q'. It should be noticed that f is orientation reversing on U and on all the characteristic suborbifolds. We remove U and U' from M and M' respectively and glue their boundaries through the diffeomorphism f restricted to the boundaries. The resulting space is a torus orbifold with the orientations compatible with the torus orbifolds M and M'.

Let us explain the equivariant connected sum along characteristic suborbifolds. For the sake of simplicity we assume that $d(M) = 1$. Let M_i be a characteristic suborbifold, p a point in the principal stratum of the orbifold M. We may assume that the isotropy subgroup at p of the T-action is the circle group S_i. Let $\tilde{S}_i$ be the covering group of S_i corresponding to the edge vector v_i as introduced in the previous section. Denote by V_i the standard complex 1-dimensional $\tilde{S}_i$-module and by $D(V_i)$ the unit disk of V_i. Then it follows from the Slice Theorem and Lemma 12.3 that the T-orbit of p has a closed invariant tubular neighborhood $\bar{U}_i$ in M equivariantly diffeomorphic to

$$(T \times_{\tilde{S}_i} D(V_i)) \times D^{n-1} \tag{13.1}$$

where $T \times_{\tilde{S}_i} D(V_i)$ denotes the orbit space of $T \times D(V_i)$ by the $\tilde{S}_i$-action defined by $s(t, x) = (t\rho_i(s)^{-1}, sx)$ for $s \in \tilde{S}_i, t \in T$ and $x \in D(V_i)$.

Suppose that there are characteristic suborbifolds M_i and $M'_{i'}$ of torus orbifolds M and M' with $d(M) = d(M') = 1$ respectively such that the corresponding edge vectors coincide. Then the corresponding circle subgroups $\tilde{S}_i$ and $\tilde{S}'_{i'}$ agree and there is an equivariant diffeomorpism between $\bar{U}_i$ and $\bar{U}'_{i'}$ reversing the orientations induced from M, M_i, M' and $M'_{i'}$ because both $\bar{U}_i$ and $\bar{U}'_{i'}$ are equivariantly diffeomorphic to the space in (13.1) and D^{n-1} $(n \geq 2)$ has an orientation reversing self-diffeomorphism. We remove the interior of $\bar{U}_i$ and $\bar{U}'_{i'}$ from M and M' and paste them together along the boundaries of $\bar{U}_i$ and $\bar{U}'_{i'}$ through the orientation reversing equivariant diffeomorphism restricted to the boundaries, producing a new torus orbifold, say M''. We call this procedure the equivariant connected sum of M and M' along M_i and $M'_{i'}$. The codimension of the principal orbits in M_i and $M'_{i'}$ is $n-1$, so when $n \geq 3$, M_i and $M'_{i'}$ are pasted together to become one characteristic suborbifold in M'' and $\Delta(M'')$ is obtained from $\Delta(M)$ and $\Delta(M')$ by identifying i with i'. However, when $n = 2$, the characteristic suborbifolds M_i and $M'_{i'}$ are S^2 and the principal orbits in them are circles; so the orbits separate M_i and $M'_{i'}$ into two connected components respectively and hence two characteristic suborbifolds of M'' are produced.

Let $I \in \Sigma(M)^{(n)}$ and $I' \in \Sigma(M')^{(n)}$ be such that $C(M)(I) = C(M)(I')$. Suppose that the corresponding edge vectors are the same for I and I'. Then one can make equivariant connected sum of M and M' along each pair of characteristic suborbifolds M_i and $M'_{i'}$ such that $C(M)(i) = C(M')(i')$ for $i \in I$ and $i' \in I'$, and then elements in I and I' will be identified in pairs in the multi-fan of the resulting torus orbifold and the weights $w^{\pm}$ on the identified n-dimensional cone is the sum of those at I and I'.

We say that Δ is *connected* if Σ is connected. According to the decomposition of Σ into connected components, the multi-fan Δ decomposes into connected multi-fans which are again complete simplicial and of dimension n.

Lemma 13.4. *Suppose $n \geq 2$. Then the multi-fan Δ is geometrically realized if all connected components of Δ are geometrically realized.*

Proof. Let M be a torus orbifold of dimension $2n$ and let p be a point in the principal stratum of M. We may suppose that $d(M) = 1$. A closed tubular neighborhood $\bar{U}$ of the orbit of p is equivariantly diffeomorphic to $T \times D^n$ and the complement of $\bar{U}$ is connected because M is connected and the orbit has codimension $n \geq 2$.

Let M' be another torus orbifold of dimension $2n$ with $d(M') = 1$, and let $\bar{U}'$ be a closed subset in M' corresponding to $\bar{U}$ in M. Since both $\bar{U}$ and $\bar{U}'$ are equivariantly diffeomorphic to $T \times D^n$ and D^n has an orientation reversing diffeomorphim, there is an orientation reversing equivariant diffeomorphism between $\bar{U}$ and $\bar{U}'$. We remove the interior of $\bar{U}$ and $\bar{U}'$ from M and M' respectively and glue their boundaries through the diffeomorphism restricted to the boundaries and obtain a new torus orbifold M''. The multi-fan $\Delta(M'')$ is the disjoint union of $\Delta(M)$ and $\Delta(M')$. (Precisely speaking, $\Sigma(M'')$ is the disjoint union of $\Sigma(M)$ and $\Sigma(M')$ with the empty sets in them identified.)

If all connected components of Δ are geometrically realized, then we connect torus orbifolds that realize the connected components of Δ by the above method. Then the resulting torus orbifold realizes Δ. □

As is shown in the proof of Lemma 13.4, whenever we have more than two torus orbifolds of dimension $n \geq 2$, we can connect them and the multi-fan of the resulting torus orbifold is the disjoint union of the multi-fans of the torus orbifolds we had.

Definition. We say that a complete simplicial multi-fan $\Delta = (\Sigma, C, w^{\pm})$ of dimension n is *minimal* if

(1) Σ is isomorphic to the argumented simplicial set obtained from the boundary of an n-simplex, and
(2) the set $\{w^+(I), w^-(I)\}$ is independent of $I \in \Sigma^{(n)}$.

Although the set $\{w^+(I), w^-(I)\}$ is independent of I for a minimal multi-fan Δ, the pair $(w^+(I), w^-(I))$ may not be independent of $I \in \Sigma^{(n)}$. But one can convert Δ into another minimal multi-fan $\bar{\Delta} = (\Sigma, \bar{C}, \bar{w}^{\pm})$ such that the pair $(\bar{w}^+(I), \bar{w}^-(I))$ is independent of I.

The definition of $\bar{\Delta}$ is as follows. Since Δ is of dimension n and the cardinality of $\Sigma^{(1)}$ is $n+1$, there is a relation $\sum_{i\in\Sigma^{(1)}} b_i v_i = 0$ among the edge vectors v_i with non-zero real numbers b_i. We then define

$$\bar{C}(i) := \begin{cases} C(i) & \text{if } b_i > 0, \\ -C(i) & \text{if } b_i < 0, \end{cases}$$

and define $\bar{C}(K)$ for $K \in \Sigma^{(m)}$ with $m \geq 2$ to be the cone spanned by $\bar{C}(k)$'s for $k \in K$. We also define

$$(\bar{w}^+(I), \bar{w}^-(I)) := \begin{cases} (w^+(I), w^-(I)) & \text{if } \sharp\{i \in I \mid b_i < 0\} \text{ is even}, \\ (w^-(I), w^+(I)) & \text{if } \sharp\{i \in I \mid b_i < 0\} \text{ is odd}, \end{cases}$$

for $I \in \Sigma^{(n)}$.

Lemma 13.5. *$\bar{\Delta}$ is minimal and satisfies the following two conditions:*

(1) *the n-dimensional cones $\bar{C}(I)$ ($I \in \Sigma^{(n)}$) do not overlap and their union covers the entire space $N_{\mathbb{R}}$, and*
(2) *the pair $(\bar{w}^+(I), \bar{w}^-(I))$ is independent of $I \in \Sigma^{(n)}$.*

Moreover Δ is geometrically realized if and only if so is $\bar{\Delta}$.

Proof. Let $\bar{v}_i$ be a non-zero vector in the cone $\bar{C}(i)$. One may choose it to be v_i if $b_i > 0$ and $-v_i$ if $b_i < 0$. Then one has a relation $\sum_{i\in\Sigma^{(1)}} \bar{b}_i\bar{v}_i = 0$ with positive numbers $\bar{b}_i$. This implies the statement (1) in the lemma.

We shall prove the statement (2) in the lemma. Let $J \in \Sigma^{(n-1)}$. Since the cardinality of $\Sigma^{(1)}$ is $n+1$, there are exactly two elements $i, i' \in \Sigma^{(1)}$ not contained in J, and $J \cup \{i\}$ and $J \cup \{i'\}$ are in $\Sigma^{(n)}$, in other words, the $(n-1)$-dimensional cone $C(J)$ is a facet of only two n-dimensional cones $C(J\cup\{i\})$ and $C(J\cup\{i'\})$. We project them on $N_{\mathbb{R}}^{C(J)}$ (the quotient space of $N_{\mathbb{R}}$ by the subspace generated by $C(J)$). Then the vectors projected from v_i and $v_{i'}$ are toward opposite directions if and only if $b_i b_{i'} > 0$. It follows from the completeness of Δ that $w(J \cup \{i\}) = \operatorname{sign}(b_i b_{i'}) w(J \cup \{i'\})$. This together with the definition of $\bar{w}^{\pm}$ shows that $\bar{w}(J \cup \{i\}) = \bar{w}(J \cup \{i'\})$. Since $J \in \Sigma^{(n-1)}$ is arbitrary, this proves the statement (2). It also proves the completeness of $\bar{\Delta}$, so that $\bar{\Delta}$ is minimal.

The procedure from Δ to $\bar{\Delta}$ corresponds to reversing orientations on characteristic suborbifolds M_i with $b_i < 0$, so the latter statement in the lemma is obvious. □

Lemma 13.6. *Let Δ be a minimal multi-fan of dimension $n \geq 2$. If $n \geq 3$, then Δ is geometrically realized. If $n = 2$, then Δ is geometrically realized if (and only if) $\{w^+(I), w^-(I)\} = \{1, 0\}$ for any $I \in \Sigma^{(2)}$. In any case we can take an orbifold structure on the realizing torus orbifold such that the corresponding edge vectors $\{v_i\}$ are all primitive; that is, if $v_i = a_i v_i'$ for some $v_i' \in N$ and $a_i \in \mathbb{Z}$, then $a_i = \pm 1$.*

Proof. By Lemma 13.5, we may assume that the union of cones $C(I)$ over $I \in \Sigma^{(n)}$ covers the entire space $N_{\mathbb{R}}$ and the pair $(w^+(I), w^-(I))$, which we denote by (p, q), is independent of I. When $(p, q) = (1, 0)$, Δ can be realized by a weighted projective space, say X. There is an orbifold structure on a weighted projective space such that the edge vectors are all primitive. We admit these facts for a moment; the proof will be give in the appendix at the end of this section. Then $-X$ realizes the case when $(p, q) = (0, 1)$. This completes the proof when $n = 2$.

Suppose $n \geq 3$. For a general value of (p, q), we prepare p copies of X and q copies of $-X$ and do equivariant connected sum along all X_i's and $-X_i$'s for each $i \in \Sigma(X)$.

Then the resulting torus orbifold realizes Δ. The edge vectors are all primitive in this construction since it is so for X. □

Now let Δ be an arbitrary complete simplicial multi-fan of dimension $n \geq 2$. We decompose Δ into a number of minimal multi-fans as follows. We choose and fix a generic (rational) 1-dimensional cone in $N_{\mathbb{R}}$, say ℓ, which is not contained in any subspaces spanned by cones of dimension $\leq n-1$ in Δ. We label ℓ as $\star$. To each n-dimensional cone $C(I)$ for $I \in \Sigma^{(n)}$, we form n cones which are respectively spanned by ℓ and facets of $C(I)$. These n cones together with $C(I)$ determine a simplicial multi-fan $\Delta[I] = (\Sigma[I], C[I], w[I]^{\pm})$, where $\Sigma[I]$ consists of all proper subsets of $I \cup \{\star\}$. The weight functions $w[I]^{\pm}$ are defined as follows. Let v_i be a non-zero vector in $C(i)$ for each $i \in I$ and $v_\star$ a non-zero vector in ℓ. Then there is a relation

$$v_\star + \sum_{i \in I} a_i v_i = 0 \tag{13.2}$$

with non-zero real numbers a_i's. Let $\mathcal{I} \in \Sigma[I]^{(n)}$. Then $\mathcal{I} = I$ or $(I \backslash \{i\}) \cup \{\star\}$ for $i \in I$. We define

$$(w[I]^+(\mathcal{I}), w[I]^-(\mathcal{I})) := \begin{cases} (w^+(I), w^-(I)) & \text{if } \mathcal{I} = I \text{ or} \\ & \mathcal{I} = (I \backslash \{i\}) \cup \{\star\} \text{ and } a_i > 0, \\ (w^-(I), w^+(I)) & \text{if } \mathcal{I} = (I \backslash \{i\}) \cup \{\star\} \text{ and } a_i < 0. \end{cases} \tag{13.3}$$

Lemma 13.7. $\Delta[I]$ *is complete and hence minimal.*

Proof. The proof is essentially the same as that of lemma 13.5. As remarked in Section 2, it suffices to show that, when a generic vector v gets across an $(n-1)$-dimensional cone, the integer d_v in Section 2 remains unchanged. Let $\mathcal{J}$ be an element of $\Sigma[I]^{(n-1)}$ and let i and i' be the two elements in $(I \cup \{\star\}) \backslash \mathcal{J}$. Then $\mathcal{I} := \mathcal{J} \cup \{i\}$ and $\mathcal{I}' := \mathcal{J} \cup \{i'\}$ are the elements in $\Sigma[I]^{(n)}$ which contain $\mathcal{J}$. We project cones $C[I](\mathcal{I})$ and $C[I](\mathcal{I}')$ on $N_{\mathbb{R}}^{C[I](\mathcal{J})}$. Then it follows from (13.2) that the vectors projected from v_i and $v_{i'}$ are toward opposite directions if and only if $a_i a_{i'} > 0$, where $a_\star$ is understood to be 1. This together with the definition (13.3) of $w[I]^{\pm}$ implies that d_v remains unchanged regardless of the sign of $a_i a_{i'}$ when v gets across the $(n-1)$-dimensional cone $C[I](\mathcal{J})$. □

Let $J \in \Sigma^{(n-1)}$ and let $I_1, \dots, I_r$ be the elements in $\Sigma^{(n)}$ containing J. The n-dimensional cone spanned by $C(J)$ and ℓ appears in $\Delta[I_k]$ for $k = 1, 2 \dots, r$ with the form $C[I_k](J \cup \{\star\})$.

Lemma 13.8. $\sum_{k=1}^{r} w[I_k](J \cup \{\star\}) = 0.$

Proof. Consider the projection of the cones $C(I_k)$'s on $N_{\mathbb{R}}^{C(J)}$. We define $\operatorname{sign}(I_k) = 1$ or -1 according as the projection image of $C(I_k)$ disagrees or agrees with that of ℓ. Applying (13.3) with $I = I_k$ and $I \backslash \{i\} = J$, one sees that

$$w[I_k](J \cup \{\star\}) = \operatorname{sign}(I_k) w(I_k).$$

On the other hand, it follows from the completeness of Δ that

$$\sum_{\operatorname{sign}(I_s)=1} w(I_s) = \sum_{\operatorname{sign}(I_t)=-1} w(I_t).$$

These two identities imply the lemma. □

Proof of Theorem 13.3. By lemma 13.4 we may assume that Δ is connected. We choose a generic (rational) 1-dimensional cone ℓ and decompose Δ using ℓ into minimal multi-fans $\Delta[I]$'s ($I \in \Sigma^{(n)}$). By Lemma 13.6 $\Delta[I]$ is realized by a torus orbifold, say $M[I]$, such that all its edge vectors are primitive. We consider the disjoint union of $M[I]$ over $I \in \Sigma^{(n)}$ and piece them together using equivariant connected sum in the following way. For each $i \in \Sigma^{(1)}$ we do equivariant connected sum of $\{M[I] \mid i \in I\}$ successively along $M[I]_i$'s, and similarly do equivariant connected sum of all $M[I]$'s along $M[I]_\star$ as well. The resulting space is connected because Δ is connected, and becomes a torus orbifold. Its multi-fan is close to Δ but contains extra cones which are the cones spanned by ℓ and $C(J)$ for $J \in \Sigma^{(m)}$ with $m \leq n-1$. For a fixed $J \in \Sigma^{(n-1)}$, it follows from Lemma 13.8 that there are the same number of T-fixed points p with $\epsilon_p = 1$ and q with $\epsilon_q = -1$ contained in the union of $M[I_k]$ with $J \subset I_k$ and corresponding to the cone spanned by ℓ and $C(J)$. Hence one can do equivariant connected sum at pairs of T-fixed points p and q so that those T-fixed points will be eliminated. Doing this for each $J \in \Sigma^{(n-1)}$, we obtain a torus orbifold, say M, realizing Δ. In fact, the characteristic suborbifolds $M[I]_\star$ turn into a codimension two suborbifold of M, which is fixed by the circle subgroup determined by ℓ but has no T-fixed point, so it is not a characteristic suborbifold of M by definition. This means that all the cones in $\Delta[I]$'s containing ℓ as an edge do not show up in the mulit-fan of M. □

Proof of Theorem 13.2. We already observed the "only if" part, so we prove the "if" part. By Lemma 13.4 we may assume that our Δ, which satisfies the conditions (1) and (2) in Theorem 13.2, is connected. Then (the realization of) Σ is either

Case 1. a 1-simplex, or

Case 2. the boundary of a d-gon where $d \geq 3$,

and that

$$\{w^+(I), w^-(I)\} = \begin{cases} \{1,1\} & \text{in Case 1,} \\ \{1,0\} & \text{in Case 2.} \end{cases}$$

Using the latter statement in Lemma 13.6, the same argument as in the proof of Theorem 13.3 shows that Δ in Case 2 is geometrically realized. As for Case 1, let $I \in \Sigma^{(2)}$ be the unique simplex. There exist a finite covering $\tilde{T} \to T$ whose kernel H is isomorphic to N/N_I where N_I is the sublattice generated by the primitive vectors v_i's for $i \in I$, and a 2-dimensional $\tilde{T}$-module V corresponding to the cone $C(I)$, as was explained in Section 12. Then the one point compactification of V/H, i.e., the orbit space of S^4 by an action of N/N_I, realizes our Δ in Case 1. □

Appendix. Realization of minimal multi-fans by weighted projective spaces.

We identify the $(n+1)$-dimensional torus $T^{n+1} = S^1 \times \cdots \times S^1$ with the standard maximal torus of $GL(n+1, \mathbb{C})$ consisting of diagonal matrices. We set $\tilde{T} = T^{n+1}/D$ where D denotes the subgroup of diagonal elements $(z, \dots, z)$. It is a maximal torus in $PGL(n+1, \mathbb{C})$ and acts effectively on the projective space $\mathbb{P}^n$. Let $\tilde{S}_i$ denote the i-th factor of T^{n+1}. It is mapped injectively into $\tilde{T}$. We shall denote by the same letter $\tilde{S}_i$ its image in $\tilde{T}$. We set $\tilde{M}_i = \{[z_0, \dots, z_n] \mid z_i = 0\}$, for $i = 0, \dots, n$. They are the characteristic submanifolds of $\mathbb{P}^n$ regarded as a torus manifold with the orientations induced from the complex structure. If H is a finite subgroup of $\tilde{T}$, then the quotient $M_H = \mathbb{P}^n/H$ is a torus orbifold acted on by $T = \tilde{T}/H$ for which $(M_H, \mathbb{P}^n, H, p)$ is an orbifold chart, where $p : \mathbb{P}^n \to M_H$ is the projection. It is called a weighted projective space. Its characteristic suborbifolds are $M_i = p(\tilde{M}_i)$, $i = 0, \dots, n$, and the corresponding circle subgroups are $S_i = \pi(\tilde{S}_i)$, where $\pi : \tilde{T} \to T$ is the projection. The symmetric group $\mathcal{S}_{n+1}$ of degree $n+1$ acts on T^{n+1} and also induces an action on $\tilde{T}$. It also acts on $\mathbb{P}^n$. If H^σ denotes the transform of H by an element $\sigma \in \mathcal{S}_{n+1}$, then the transformation $\sigma : \mathbb{P}^n \to \mathbb{P}^n$ induces an isomorphism of torus manifolds $M_H \to M_{H^\sigma}$. We set

$$\mathcal{WP} = \{H \mid \text{finite subgroup of } \tilde{T}\}/\mathcal{S}_{n+1}.$$

Every element in $\mathcal{WP}$ represents an isomorphism class of weighted projective spaces.

In order to describe the multi-fan Δ_H associated with the torus orbifold M_H we introduce the following notations:

$$\tilde{N} = \mathbb{Z}^{n+1}/\text{diagonal submodule}, \quad \tilde{v}_i = \text{image of } e_i \text{ in } \tilde{N}, \quad N = \mathbb{Z}^n,$$

where e_i is the i-th fundamental unit vector in $\mathbb{Z}^{n+1}$. $\tilde{N}$ is canonically identified with $\mathrm{Hom}(S^1, \tilde{T})$. If one chooses an identification of $\mathrm{Hom}(S^1, T) = H_2(BT; \mathbb{Z})$ with N, then the finite covering map $\pi : \tilde{T} \to T$ induces an injective homomorphism $\varphi : \tilde{N} \to N$. The vectors $v_i = \varphi(\tilde{v}_i)$ are the edge vectors of the 1-dimensional cones of Δ_H. Note that they satisfy the equality

$$\sum_i v_i = 0, \tag{13.4}$$

since the $\tilde{v}_i$'s satisfy a similar equality. This implies that Δ_H is a minimal multi-fan satisfying the conditon (1) in Lemma 13.5. It is also clear that $(w^+(I), w^-(I)) = (1, 0)$. We shall denote by $\mathcal{MF}$ the set of minimal multi-fans satisfying the above two conditions. If one chooses another identification of $\mathrm{Hom}(S^1, T)$ with N, then φ is transformed to $\psi \circ \varphi$ where $\psi \in GL(n, \mathbb{Z})$. $GL(n, \mathbb{Z})$ acts on $\mathcal{MF}$ from left by transforming the cones simultaneously by its elements. Let $d_H \in \mathbb{Z}$ be the maximal common divisor of the edge vectors v_i of Δ_H. We get a correspondence

$$\alpha : \mathcal{WP}/\mathcal{S}_{n+1} \to GL(n, \mathbb{Z})\backslash \mathcal{MF} \times \mathbb{Z}_{>0}$$

which sends H to (Δ_H, d_H).

Lemma 13.9. *The correspondence α is a bijection. In particular, every minimal multi-fan Δ in $\mathcal{MF}$ is realizable.*

Proof. We shall define a correspondence $\beta : GL(n, \mathbb{Z})\backslash \mathcal{MF} \times \mathbb{Z}_{>0} \to \mathcal{WP}/\mathcal{S}_{n+1}$ which is to be the inverse of α. Take a multi-fan Δ in $\mathcal{MF}$ and $d \in \mathbb{Z}_{>0}$. It is easy to see there is a unique set $\{v_i\}$ of edge vectors of Δ such that $\sum_i v_i = 0$ and the maximal common divisor of $\{v_i\}$ is d. Define a homomorphism $\varphi : \tilde{N} \to N$ by requiring $\varphi(\tilde{v}_i) = v_i$. Then there is a unique finite covering map $\pi : \tilde{T} \to T$ which induces $\varphi : \tilde{N} = \mathrm{Hom}(S^1, \tilde{T}) \to N = \mathrm{Hom}(S^1, T)$. Let H be the kernel of π. The homomorphism φ, hence H either, does not depend on the choice of identification $N = \mathrm{Hom}(S^1, T)$, but it depends on the numbering of v_i's. So if we put $\beta(\Delta, d) =$ the class of H in $\mathcal{WP}/\mathcal{S}_{n+1}$, it induces a correspondence β as above. It is clear that β is in fact the inverse of α. □

Remark. Let a be a positive integer. The correspondence $T^{n+1} \ni (z_0, z_1, \dots, z_n) \mapsto (z_0^a, z_1^a, \dots, z_n^a) \in T^{n+1}$ induces a homomorphism $\rho : \tilde{T} \to \tilde{T}$. For a finite group H of $\tilde{T}$ define $H' = \rho^{-1}(H)$. The edge vectors $\{v_i'\}$ corresponding to the torus manifold $M_{H'}$ are of the form $v_i' = a v_i$, where $\{v_i\}$ correspond to M_H. Hence $\Delta_H = \Delta_{H'}$ and $d_{H'} = a d_H$. Let

$g : \mathbb{P}^n \to \mathbb{P}^n$ be the map defined by $g[z_0, z_1, \ldots, z_n] = [z_0^a, z_1^a, \ldots, z_n^a]$. Then it induces a homeomorphism $M_{H'} \to M_H$ which is equivariant with respect to the isomorphism of tori between $\tilde{T}/H'$ and $\tilde{T}/H$ induced by ρ. If M_H and $M_{H'}$ are considered as algebraic varieties then the homeomorphism becomes an equivalence. It is a fundamental fact in the theory of toric varieties that to each fan corresponds a toric variety. The above equivalence gives an interpretation of this fact within this special case in our context. Related results are found in [22]. Related to the above remark, for a later use, we point out the following fact. Let $a_i, \ldots, a_n$ be positive integers, and let $\mathbb{Z}/a_i \subset S^1$ be the subgroup of a_i-th roots of unity. Set $G = \prod_i \mathbb{Z}/a_i$. Then the map $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto (z_1^{a_1} \ldots, z_n^{a_n}) \in \mathbb{C}^n$ induces an equivalence of affine algebraic varieties $\mathbb{C}^n/G \to \mathbb{C}^n$.

Let $M_H \in \mathcal{WP}$ and let $\{v_i\}$ be the edge vectors corresponding to the orbifold structure as given above. Even if $d_H = 1$, it may happen that some of v_i's are not primitive. We will show that there always exists a torus orbifold structure on M_H such that the corresponding edge vectors are all primitive. More generally we have

Lemma 13.10. *Let M_H be a weighted projective space and $\{v_i\}$ the corresponding edge vectors satisfying $\sum_i v_i = 0$ as given above. Suppose that $\{v_i'\}$ are vectors in N such that $v_i = a_i v_i'$ with $a_i \in \mathbb{Z}_{>0}$. Then there is an orbifold structure on M_H which admits $\{v_i'\}$ as the corresponding edge vectors.*

Proof. For each $x \in M_H$ let $\tilde{T}_x \subset \tilde{T}$ be the isotropy subgroup at $\tilde{x}$ of the $\tilde{T}$-action on $\mathbb{P}^n$ where $\tilde{x} \in p^{-1}(x)$. $\tilde{T}_x$ does not depend on the choice of $\tilde{x}$ in $p^{-1}(x)$. If x lies in $\operatorname{Int} M_I = M_I \setminus \bigcup_{J \supset I} M_J$ for $I \in \Sigma(M_H)^{(k)}$, then $\tilde{T}_x = \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We put $H_x = H \cap \tilde{T}_x$. We take a family $\{V_{x,\mu} | \mu \in \mathbb{Z}_{>0}\}$ of small $\tilde{T}_x$-invariant open neighborhoods of $\tilde{x}$ such that $V_{x,\mu}$ converges to $\tilde{x}$ when μ tends to infinity. We may assume that $V_{x,\mu}$ is equivariantly diffeomorphic to an $\tilde{S}_I$-invariant open disk in $\mathbb{C}^n$. It is possible to make $V_{x,\mu}$'s so small that they satisfy the following condition:

$$H_x = \{h \in H \mid h \cdot V_{x,\mu} \cap V_{x,\mu} \neq \emptyset\}. \tag{13.5}$$

Then $U_{x,\mu} = V_{x,\mu}/H_x$ is an open neighborhood of $\tilde{x}$ in M_H, and $(U_{x,\mu}, V_{x,\mu}, H_x, p|V_{x,\mu})$ is an orbifold chart of M_H compatible with $(M_H, \mathbb{P}^n, H, p)$.

On the other hand the fact that $v_i = a_i v_i'$ implies that the kernel of $p : \tilde{S}_i \to S_i$ contains $\mathbb{Z}/a_i$,which we denote by G_i. Since H is the kernel of $p : \tilde{T} \to T$, G_i is contained in H. We put $G_I = \prod_{I \in I} G_i$ for $I \in \Sigma(M_H)^{(k)}$ and define

$$V_{x,\mu}' = V_{x,\mu}/G_I, \quad H_x' = H_x/G_I \quad \text{for } x \in \operatorname{Int} M_I.$$

$V_{x,\mu}'$ can be considered as an open disk in $\mathbb{C}^n$ as pointed out in Remark above. The projection $p|V_x : V_x \to U_x$ induces a map $p_{x,\mu}' : V_{x,\mu}' \to U_x$ which induces a homeomorphism $V_{x,\mu}'/H_x' \to U_x$.

We shall prove that the family $\{(U_{x,\mu}', V_{x,\mu}', H_x', p_{x,\mu}') \mid x \in M, \ \mu \in \mathbb{Z}_{>0}\}$ forms a set of orbifold charts of an orbifold structure on M_H. For that purpose it suffices to show that, if $U_{x.\mu}' \subset U_{y,\nu}'$, then there are an open embedding $\phi : V_{x,\mu}' \to V_{y,\nu}'$ and an injective homomorphism $\rho : H_x' \to H_y'$ such that

$$\rho(H_x') = \{h \in H_y' \mid h \cdot \phi(V_{x,\mu}') \cap \phi(V_{x,\mu}') \neq \emptyset\}. \tag{13.6}$$

The condition (13.5) implies that, if $x \in \operatorname{Int} M_I$ and $y \in \operatorname{Int} M_J$ with I and $J \in \Sigma(M_H)$, and if $U_{x.\mu}' \subset U_{y,\nu}'$, then $I \supset J$. Therefore

$$H_x \subset H_y \quad \text{and} \quad G_I \cap H_y = G_J.$$

It follows that the inclusion $H_x \to H_y$ induces an injective homomorphism $\rho : H'_x = H_x/G_I \to H_y/G_J = H'_y$. If $\tilde{x}$ is taken in $V_{y,\nu}$, then $V_{x,\mu}$ is contained in $V_{y,\nu}$. The inclusion induces an embedding $\phi : V'_{x,\mu} \to V'_{y,\nu}$. The condition (13.6) follows from (13.5).

If x lies in M_i, then the action of S_i lifts to the action of $\tilde{S}'_i = \tilde{S}_i/G_i$ and the lifting is minimal. Hence the edge vector of $C(i)$ corresponding to the orbifold structure defined above must be v'_i. □

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