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A TWINING CHARACTER FORMULA FOR DEMAZURE MODULES

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To Professor Yukio Tsushima on the occasion of his sixtieth birthday

Abstract. We prove a formula for the twining characters of certain Demazure modules, over a Borel subalgebra $\mathfrak{b}$ of a finite-dimensional complex semi-simple Lie algebra $\mathfrak{g}$. This formula describes the twining character of the Demazure module by the ω-Demazure operator associated to an element of the Weyl group that is fixed by the Dynkin diagram automorphism ω of $\mathfrak{g}$. Our result is a refinement of the twining character formula for the irreducible highest weight $\mathfrak{g}$-modules of symmetric dominant integral highest weights, and also of the ordinary Demazure character formula.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose a set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group.

A bijection ω of the index set I such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$ induces an automorphism ω (see §1.1), called a (Dynkin) diagram automorphism, of the Lie algebra $\mathfrak{g}$, which stabilizes $\mathfrak{h}_{\mathbb{Z}} = \sum_{i \in I} \mathbb{Z} h_i$. Note that this bijection ω of I also induces an automorphism ${}^t\omega$ of the dual Lie algebra ${}^t\mathfrak{g}$ of $\mathfrak{g}$ in a similar way, where the dual Lie algebra ${}^t\mathfrak{g}$ is a complex semi-simple Lie algebra with the Dynkin diagram opposite to the one for $\mathfrak{g}$. We denote by $\langle \omega \rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(\mathfrak{g})$ generated by the diagram automorphism ω. The restriction of ω to $\mathfrak{h}$ induces a transposed map $\omega^* \colon \mathfrak{h}^* \to \mathfrak{h}^*$, which stabilizes the integral weight lattice $\mathfrak{h}_{\mathbb{Z}}^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\} \simeq \mathbf{Ab}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. We set $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$, $W^\omega = \{w \in W \mid \omega^* w = w \omega^*\}$, $(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \simeq (\mathfrak{h}^0)^*$, and $(\mathfrak{h}_{\mathbb{Z}}^*)^0 = \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid \omega^*(\lambda) = \lambda\}$.

In [FSS] and [FRS], they introduced a certain Lie algebra $\widehat{\mathfrak{g}}$, called the orbit Lie algebra, which is nothing but the dual Lie algebra ${}^t\big(({}^t\mathfrak{g})^{{}^t\omega}\big)$ of the (semi-simple) fixed

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point subalgebra $({}^t\mathfrak{g})^{{}^t\omega}$ of ${}^t\mathfrak{g}$ by the automorphism ${}^t\omega$ of ${}^t\mathfrak{g}$. Let $\widehat{\mathfrak{h}}$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$ the Borel subalgebra, and $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ the set of positive roots chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$. Let $\{\widehat{\alpha}_i \mid i \in \widehat{I}\}$ be the set of simple roots in $\widehat{\Delta}_+$, $\{\widehat{h}_i \mid i \in \widehat{I}\}$ the set of simple coroots in $\widehat{\mathfrak{h}}$, and $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I}\rangle \subset GL(\widehat{\mathfrak{h}}^*)$ the Weyl group, where the index set $\widehat{I}$ is a set of representatives of the ω-orbits in I. It is well-known that there exist an isomorphism of groups $\Theta \colon \widehat{W} \to W^\omega$ and a $\mathbb{C}$-linear isomorphism $P_\omega : \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ such that if $P_\omega^* \colon \widehat{\mathfrak{h}}^* \tilde{\to} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ is the transposed map of P_ω, then $\Theta(\widehat{w})|_{(\mathfrak{h}^*)^0} = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for all $\widehat{w} \in \widehat{W}$. We set $w_i = \Theta(\widehat{r}_i) \in W^\omega$ for $i \in \widehat{I}$. In particular, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system.

For dominant $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$, let $L(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight λ. It admits a unique $\mathbb{C}$-linear $\langle\omega\rangle$-action such that $\omega \cdot (xv) = \omega(x)(\omega \cdot v)$ for each $x \in \mathfrak{g}$, $v \in L(\lambda)$, and such that $\omega \cdot v_\lambda = v_\lambda$, where v_λ is a (nonzero) highest weight vector of $L(\lambda)$. So therefore does its dual module $L(\lambda)^* \simeq L(-w_0(\lambda))$ with w_0 the longest element in W. In [FSS] and [FRS], they defined the twining character $\mathrm{ch}^\omega(L(\lambda))$ of $L(\lambda)$ by

$$\mathrm{ch}^\omega(L(\lambda)) = \sum_{\mu \in (\mathfrak{h}_\mathbb{Z}^*)^0} \mathrm{Tr}(\omega|_{L(\lambda)_\mu})\, e(\mu)$$

in the group algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ over $\mathbb{C}$ of $(\mathfrak{h}_\mathbb{Z}^*)^0$ with basis $e(\mu)$, $\mu \in (\mathfrak{h}_\mathbb{Z}^*)^0$, and they proved

$$\mathrm{ch}^\omega(L(\lambda)) = P_\omega^*\Big(\mathrm{ch}\, \widehat{L}(\widehat{\lambda})\Big),$$

where $\mathrm{ch}\, \widehat{L}(\widehat{\lambda}) \in \mathbb{C}[\widehat{\mathfrak{h}}^*]$ is the ordinary character of the simple $\widehat{\mathfrak{g}}$-module $\widehat{L}(\widehat{\lambda})$ of dominant integral highest weight $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$.

Let $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$, and for each $w \in W^\omega$, let $J_w(\lambda) = \mathfrak{U}(\mathfrak{b})v_{w(\lambda)}^* \subset L(\lambda)^*$ be Joseph's module of highest weight $-w(\lambda)$ in $L(\lambda)^*$, with $v_{w(\lambda)}^*$ a (nonzero) weight vector in $L(\lambda)^*$ of weight $-w(\lambda)$. In this paper we will prove a formula of Demazure type for the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of $J_w(\lambda)$ defined by

$$\mathrm{ch}^\omega(J_w(\lambda)) = \sum_{\mu \in (\mathfrak{h}_\mathbb{Z}^*)^0} \mathrm{Tr}(\omega|_{J_w(\lambda)_\mu})\, e(\mu).$$

As a corollary, we will find a striking relation:

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\Big(\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda})\Big),$$

where $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda}) \in \mathbb{C}[\widehat{\mathfrak{h}}^*]$ is the ordinary character of Joseph's module $\widehat{J}_{\widehat{w}}(\widehat{\lambda})$ of highest weight $-\widehat{w}(\widehat{\lambda})$ for the orbit Lie algebra $\widehat{\mathfrak{g}}$.

To explain our result precisely, we need some more notation. Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. We will identify the rational character group $\Lambda = \mathbf{Grp}_\mathbb{C}(T, GL_1)$ of T with $\mathfrak{h}_\mathbb{Z}^*$. The diagram automorphism ω of $\mathfrak{g}$ lifts to an automorphism of G, which we will by abuse of notation denote by ω. We will also denote the induced action of ω on Λ by the same letter ω, and set $\Lambda^\omega = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$.

For a rational $\langle \omega \rangle \ltimes T$-module V, we define the twining character $\mathrm{ch}^\omega(V) \in \mathbb{C}[\Lambda^\omega]$ of V to be

$$\mathrm{ch}^\omega(V) = \sum_{\mu \in \Lambda^\omega} \mathrm{Tr}(\omega|_{V_\mu})\, e(\mu),$$

where V_μ is the μ-weight space of V. Here we note that the twining character $\mathrm{ch}^\omega(V) \in \mathbb{C}[\Lambda^\omega]$ can be viewed as the trace function

$$T \ni t \mapsto \mathrm{Tr}((\omega, t)\ ;\ V) \in \mathbb{C}.$$

In fact, we have for each $t \in T$,

$$\mathrm{Tr}((\omega, t)\ ;\ V) = \sum_{\mu \in \Lambda^\omega} \mathrm{Tr}(\omega|_{V_\mu})\, \mu(t) \in \mathbb{C}.$$

Fix $w \in W^\omega$, and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, which is the Zariski closure in the flag variety G/B of the Bruhat cell $B\dot{w}B/B$, where $\dot{w} \in N_G(T)$ denotes a right coset representative of $w \in W \simeq N_G(T)/T$ fixed by $\omega \in \mathrm{Aut}(G)$. If M is a rational $\langle \omega \rangle \ltimes B$-module, then the B-equivariant $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to M carries a structure of $\langle \omega \rangle \ltimes B$-equivariant sheaf (see §2.3), so that its cohomology groups $H^\bullet(X(w), \mathcal{L}_{X(w)}(M))$ are $\langle \omega \rangle \ltimes B$-modules. For each $\lambda \in \Lambda^\omega$, we let $\mathbb{C}_\lambda$ denote the one-dimensional $\langle \omega \rangle \ltimes B$-module on which B acts via λ through the quotient $B \to T$ and $\langle \omega \rangle$ trivially. We call $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for dominant $\lambda \in \Lambda^\omega$ a Demazure module. Now Joseph's module $J_w(\lambda)$ admits a structure of $\langle \omega \rangle \ltimes B$-module, and we have an isomorphism of $\langle \omega \rangle \ltimes B$-modules (see §3.2)

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)),$$

where $J_w(\lambda)^*$ is the dual $\langle \omega \rangle \ltimes B$-module of $J_w(\lambda)$. We can now state our main result in this paper.

Theorem 0.1. *Let M be a finite-dimensional rational $\langle \omega \rangle \ltimes B$-module, $w \in W^\omega$, and let $w = w_{i_1} w_{i_2} \cdots w_{i_n}$ be a reduced expression in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\sum_{j \geq 0} (-1)^j\, \mathrm{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) = \widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_n} \Big(\mathrm{ch}^\omega(M) \Big),$$

where $\widehat{D}_i$, $i \in \widehat{I}$, is the ω-Demazure operator (see §1.4). In particular, for dominant $\lambda \in \Lambda^\omega$, we have

$$\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_n}(e(\lambda)).$$

Now let $\widehat{\mathfrak{h}}_\mathbb{Z} = \sum_{i \in \widehat{I}} \mathbb{Z}\, \widehat{h}_i$ and $\widehat{\mathfrak{h}}_\mathbb{Z}^* = \mathbf{Ab}(\widehat{\mathfrak{h}}_\mathbb{Z}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$. For dominant $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, let $\widehat{L}(\widehat{\lambda})$ be the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, and for each $\widehat{w} \in \widehat{W}$, let $\widehat{J}_{\widehat{w}}(\widehat{\lambda}) = \mathfrak{U}(\widehat{\mathfrak{b}})\, \widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \subset \widehat{L}(\widehat{\lambda})^*$ be Joseph's module of highest weight $-\widehat{w}(\widehat{\lambda})$ with $\widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \in \widehat{L}(\widehat{\lambda})^*$ a (nonzero) weight vector of weight $-\widehat{w}(\widehat{\lambda})$.

Corollary 0.2. *Let* $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$ *be dominant and* $w \in W^\omega$. *We set* $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*$ *and* $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$. *Then we have in* $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$,

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\left(\mathrm{ch}\,\widehat{J}_{\widehat{w}}(\widehat{\lambda})\right),$$

where P_ω^* *is a* $\mathbb{C}$*-algebra isomorphism* $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*]\tilde{\to}\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ *defined by* $P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu}))$ *for each basis element* $e(\widehat{\mu})$, $\widehat{\mu} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*$, *of the group algebra* $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*]$ *over* $\mathbb{C}$ *of* $\widehat{\mathfrak{h}}_{\mathbb{Z}}^*$.

The paper is organized as follows. In §1 we assemble some definitions and properties of orbit Lie algebras and of twining characters needed later. In §2 we study the $\langle\omega\rangle \ltimes B$-equivariant Demazure-Hansen desingularizations of the $\langle\omega\rangle$-invariant Schubert varieties, and some $\langle\omega\rangle \ltimes B$-equivariant sheaves on these varieties. We finish the proof of our main theorem in §3, using all the materials above and the $\langle\omega\rangle \ltimes B$-equivariant Leray spectral sequences.

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Notation. In this paper we mainly follow the notation of [Ja] except that for a category $\mathcal{C}$ and its objects A and B, the symbol $\mathcal{C}(A,B)$ will denote the set of morphisms of $\mathcal{C}$ from A to B. The following is a list of symbols for the categories we will be working in:

Ab the category of abelian groups
$\mathbf{Grp}_{\mathbb{C}}$ the category of linear algebraic groups over $\mathbb{C}$
Var the category of varieties over $\mathbb{C}$
$\mathbf{Mod}_X$ the category of $\mathcal{O}_X$-modules, $\mathcal{O}_X$ the structure sheaf of a variety X

1. Twining characters

For details about diagram automorphisms and orbit Lie algebras briefly explained in §1.1 and §1.2 below, see [FSS], [FRS], and [N1]–[N3].

1.1. Diagram automorphisms

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose a set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j\in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I\rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_{\mathbb{Z}} = \sum_{i\in I} \mathbb{Z}h_i$.

We fix a bijection $\omega\colon I \to I$ of the index set I such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all } i,j \in I.$$

Let N be the order of ω, and N_i the number of elements of the ω-orbit of $i \in I$. This ω can be extended in a unique way to an automorphism (also denoted by ω) of order N of the Lie algebra $\mathfrak{g}$ in such a way that

$$\begin{cases} \omega(e_i) = e_{\omega(i)}, & i \in I, \\ \omega(f_i) = f_{\omega(i)}, & i \in I, \\ \omega(h_i) = h_{\omega(i)}, & i \in I, \end{cases}$$

where $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$ for $i \in I$ are the Chevalley generators. In a similar way, the bijection $\omega\colon I \to I$ can also be extended to an automorphism ${}^t\omega$ of the dual Lie algebra ${}^t\mathfrak{g}$ of $\mathfrak{g}$, where the dual Lie algebra ${}^t\mathfrak{g}$ is a complex semi-simple Lie algebra which has the Dynkin diagram opposite to that of $\mathfrak{g}$. Note that we have $(\omega(x)|\omega(y)) = (x|y)$ for $x, y \in \mathfrak{g}$, where $(\cdot|\cdot)$ is the suitably normalized Killing form on $\mathfrak{g}$ (cf. [N2, §3.1]), and that the restriction of ω to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$ such that $\omega^*(\lambda)(h) = \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, which is also an isometry with respect to $(\cdot|\cdot)$. We set $(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}$, $\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \quad \text{for all } i \in I\} \simeq \mathbf{Ab}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$, and $(\mathfrak{h}^*_{\mathbb{Z}})^0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\}$. Note that the Weyl vector $\rho = (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ is in $(\mathfrak{h}^*_{\mathbb{Z}})^0$.

1.2. Orbit Lie algebras

We choose and fix a set $\widehat{I}$ of representatives of the ω-orbits in I, and set $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$, where $\widehat{a}_{ij}$ is given by

$$\widehat{a}_{ij} = s_j \times \sum_{k=0}^{N_j - 1} a_{i,\omega^k(j)} \quad \text{for } i, j \in \widehat{I} \quad \text{with} \quad s_j = \frac{2}{\sum_{k=0}^{N_j-1} a_{j,\omega^k(j)}} \quad \text{for } j \in \widehat{I}.$$

Set for each $i \in \widehat{I}$, $I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$. We know from [FRS, §2] that for each $i \in \widehat{I}$,

$$\sum_{k \in I_i} a_{ik} = 1 \text{ or } 2.$$

Moreover, there are only two possibilities:

(a) if $\sum_{k \in I_i} a_{ik} = 1$, then N_i is even and the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is of type $A_2 \times \cdots \times A_2$ (where A_2 appears $N_i/2$ times);

(b) if $\sum_{k \in I_i} a_{ik} = 2$, then the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is totally disconnected and of type $A_1 \times \cdots \times A_1$ (where A_1 appears N_i times).

The orbit Lie algebra associated to the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ is defined to be the complex semi-simple Lie algebra $\widehat{\mathfrak{g}}$ associated to the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ with the Cartan subalgebra $\widehat{\mathfrak{h}}$, the Borel subalgebra $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$, the set of positive roots $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$, the set of simple roots $\{\widehat{\alpha}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}^*$, the set of simple coroots $\{\widehat{h}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}$, and the Weyl group $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I} \rangle \subset GL(\widehat{\mathfrak{h}}^*)$.

Remark 1.2.1. The Dynkin diagram of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is in general disconnected, and so the $\widehat{\mathfrak{g}}$ is a direct sum of simple Lie algebras. We can easily determine the explicit diagram of $\widehat{\mathfrak{g}}$ from the argument in [N2, §3.2], by using the table in [FSS, §2.4] for the case where $\mathfrak{g}$ is a simple Lie algebra. Moreover, by using the results of [Kac, 7.9–7.10] for the case where $\mathfrak{g}$ is simple (see also [N1, §4] for a more general case), we can easily deduce that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is nothing but the dual Lie algebra ${}^t\big(({}^t\mathfrak{g})^{{}^t\omega}\big)$ of the (semi-simple) fixed point subalgebra $({}^t\mathfrak{g})^{{}^t\omega} = \{x \in {}^t\mathfrak{g} \mid ({}^t\omega)(x) = x\}$ of ${}^t\mathfrak{g}$ by the automorphism ${}^t\omega$ of ${}^t\mathfrak{g}$.

We set $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$. Then there exists a linear isomorphism $P_\omega \colon \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ given by

$$P_\omega\left(\sum_{k\in I_i} h_k\right) = N_i\,\widehat{h}_i \quad \text{for each } i \in \widehat{I}. \tag{1}$$

Since this map $P_\omega \colon \mathfrak{h}^0 \xrightarrow{\sim} \widehat{\mathfrak{h}}$ is an isometry with respect to the respective Killing forms on $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$, it induces a transposed map (which is also isometric) $P_\omega^* \colon \widehat{\mathfrak{h}}^* \xrightarrow{\sim} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ such that $P_\omega^*(\widehat{\lambda})(h) = \widehat{\lambda}(P_\omega(h))$ for $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$, $h \in \mathfrak{h}^0$. Note that we have for each $i \in \widehat{I}$,

$$P_\omega^*(\widehat{\alpha}_i) = s_i\beta_i, \tag{2}$$

where $\beta_i = \sum_{k\in I_i}\alpha_k \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$. Also, if $\widehat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{i\in\widehat{I}} \mathbb{Z}\,\widehat{h}_i$ and $\widehat{\mathfrak{h}}_{\mathbb{Z}}^* = \mathbf{Ab}(\widehat{\mathfrak{h}}_{\mathbb{Z}}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$, then (see §1.6) $P_\omega^*(\widehat{\mathfrak{h}}_{\mathbb{Z}}^*) = (\mathfrak{h}_{\mathbb{Z}}^*)^0$.

We now define the subgroup W^ω of W by

$$W^\omega = \{w \in W \mid \omega^* w = w\omega^*\}.$$

It is well-known (see, e.g., [FRS]) that there exists an isomorphism of groups

$$\Theta \colon \widehat{W} \to W^\omega$$

from the Weyl group $\widehat{W}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ onto the group W^ω such that the following diagram commutes for each $\widehat{w} \in \widehat{W}$:

$$\begin{array}{ccc} \widehat{\mathfrak{h}}^* & \xrightarrow{P_\omega^*} & (\mathfrak{h}^*)^0 \\ {\scriptstyle \widehat{w}}\big\downarrow & & \big\downarrow {\scriptstyle \Theta(\widehat{w})|_{(\mathfrak{h}^*)^0}} \\ \widehat{\mathfrak{h}}^* & \xrightarrow[P_\omega^*]{} & (\mathfrak{h}^*)^0. \end{array} \tag{3}$$

For each $i \in \widehat{I}$, set $w_i = \Theta(\widehat{r}_i) \in W^\omega$. Explicitly,

$$w_i = \begin{cases} \displaystyle\prod_{k=0}^{N_i/2-1} \left(r_{\omega^k(i)}\, r_{\omega^{k+N_i/2}(i)}\, r_{\omega^k(i)}\right) & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \displaystyle\prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases} \tag{4}$$

Hence each w_i is the longest element of the subgroup W_{I_i} of the Weyl group W generated by the r_k's for $k \in I_i$. Notice that $w_{\omega(i)} = w_i$ and $w_i^2 = 1$ for $i \in \widehat{I}$. Furthermore, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system as $(\widehat{W}, \{\widehat{r}_i \mid i \in \widehat{I}\})$ does. We will denote the length function of the Coxeter system $(W, \{r_i \mid i \in I\})$ (resp. $(W^\omega, \{w_i \mid i \in \widehat{I}\})$) by $\ell \colon W \to \mathbb{Z}_{\geq 0}$ (resp. $\widehat{\ell} \colon W^\omega \to \mathbb{Z}_{\geq 0}$).

1.3. A Lemma about the length functions

Lemma 1.3.1. *Let $w = w_{i_1} w_{i_2} \cdots w_{i_n} \in W^\omega$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$. Then $\ell(w) = \ell(w_{i_1}) + \ell(w_{i_2}) + \cdots + \ell(w_{i_n})$.*

Proof. We argue by induction on n. For $n = 1$, the assertion is trivial. Suppose that $n \geq 2$, and set $u = w_{i_2} \cdots w_{i_n} \in W^\omega$. Because $u = w_{i_2} \cdots w_{i_n}$ is a reduced expression in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, it follows from the induction hypothesis that $\ell(u) = \ell(w_{i_2}) + \cdots + \ell(w_{i_n})$. Hence we need to show that $\ell(w) = \ell(w_{i_1}) + \ell(u)$. For this purpose, we set

$$\Delta(w) = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) \in \Delta_-\},$$

$$\Delta(u) = \{\alpha \in \Delta_+ \mid u^{-1}(\alpha) \in \Delta_-\},$$

$$\Delta(w_{i_1}) = \{\alpha \in \Delta_+ \mid w_{i_1}^{-1}(\alpha) \in \Delta_-\},$$

and show that

$$\Delta(w) \supset \Delta(w_{i_1}) \sqcup w_{i_1}(\Delta(u)), \tag{5}$$

which we leave to the reader as an easy exercise. Then the assertion that $\ell(w) = \ell(w_{i_1}) + \ell(u)$ immediately follows since

$$\begin{aligned}\ell(w) = \# \Delta(w) &\geq \# \Delta(w_{i_1}) + \# \Delta(u) \\ &= \ell(w_{i_1}) + \ell(u),\end{aligned}$$

while the reverse inequality $\ell(w) = \ell(w_{i_1} u) \leq \ell(w_{i_1}) + \ell(u)$ is obvious. □

Remark 1.3.2. It is obvious that the map $\alpha \mapsto \omega^*(\alpha)$ gives a bijection from the set $\Delta(\omega^* w (\omega^*)^{-1})$ onto the set $\Delta(w)$. Hence we see that the longest element $w_0 \in W$ belongs to W^ω. In fact, arguing as in the proof of Lemma 1.3.1, we can easily show that the isomorphism $\Theta \colon \widehat{W} \tilde{\to} W^\omega$ maps the longest element $\widehat{w}_0 \in \widehat{W}$ to the longest element $w_0 \in W$.

1.4. The ω-Demazure operators

Recall the ordinary Demazure operator D_i, $i \in I$, on the group ring $\mathbb{Z}[\mathfrak{h}_\mathbb{Z}^*] = \coprod_{\lambda \in \mathfrak{h}_\mathbb{Z}^*} \mathbb{Z} e(\lambda)$:

$$D_i \colon e(\lambda) \mapsto \frac{e(\lambda) - e(-\alpha_i) e(r_i(\lambda))}{1 - e(-\alpha_i)}.$$

Let $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ be the group algebra over $\mathbb{C}$ of $\widehat{\mathfrak{h}}_\mathbb{Z}^*$ with basis $e(\widehat{\lambda})$, $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$. Define likewise the Demazure operator $D_{\widehat{r}_i}$, $i \in \widehat{I}$, on $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ to be the $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ given by

$$D_{\widehat{r}_i}(e(\widehat{\lambda})) = \frac{e(\widehat{\lambda}) - e(-\widehat{\alpha}_i) e(\widehat{r}_i(\widehat{\lambda}))}{1 - e(-\widehat{\alpha}_i)}.$$

Then transfer $D_{\widehat{r}_i}$ via P_ω^* onto the group algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ to define the ω-Demazure operator

$$\widehat{D}_i = P_\omega^* \circ D_{\widehat{r}_i} \circ (P_\omega^*)^{-1} \quad \text{for } i \in \widehat{I}. \tag{6}$$

Thus we can easily check the following.

Lemma 1.4.1. *Let $i \in \widehat{I}$. For each $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$, we have*

$$\widehat{D}_i(e(\lambda)) = \frac{e(\lambda) - e(-s_i\beta_i)e(w_i(\lambda))}{1 - e(-s_i\beta_i)},$$

and moreover

$$\widehat{D}_i(e(\lambda)) = \begin{cases} e(\lambda) + e(\lambda - s_i\beta_i) + \cdots + e(w_i(\lambda)) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{if } \lambda(h_i) = -1, \\ -\Big(e(\lambda + s_i\beta_i) + e(\lambda + 2s_i\beta_i) + \cdots + e(w_i(\lambda + s_i\beta_i))\Big) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\leq -2}. \end{cases}$$

Remark 1.4.2. Let $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$. We set $\widehat{D}_w = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0])$. Then we have by the definition that

$$\widehat{D}_w = P_\omega^* \circ \left(D_{\widehat{r}_{i_1}}D_{\widehat{r}_{i_2}}\cdots D_{\widehat{r}_{i_n}}\right) \circ (P_\omega^*)^{-1}.$$

Hence we see that the operator $\widehat{D}_w \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0])$ defined above does not depend on the choice of the reduced expression of $w \in W^\omega$ as the product $D_{\widehat{r}_{i_1}}D_{\widehat{r}_{i_2}}\cdots D_{\widehat{r}_{i_n}} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*])$ of the ordinary Demazure operators does not depend on the choice of the reduced expression of $\Theta^{-1}(w) = \widehat{r}_{i_1}\widehat{r}_{i_2}\cdots\widehat{r}_{i_n} \in \widehat{W}$ in the Coxeter system $(\widehat{W}, \{\widehat{r}_i \mid i \in \widehat{I}\})$.

1.5. Twining characters

Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \mathbf{Grp}_{\mathbb{C}}(T, GL_1)$ of T may be identified with $\mathfrak{h}_{\mathbb{Z}}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^\vee \rangle = (\mathrm{d}\lambda)(h_i)$, where $\alpha_i^\vee \in \mathbf{Grp}_{\mathbb{C}}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. Let $\Lambda_+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad \text{for all } i \in I\}$ be the set of dominant weights of Λ.

For each root $\alpha \in \Lambda$, let $u_\alpha : \mathbf{G}_a \to G$ be a morphism defining the root subgroup of G associated to α. We choose $u_{\pm\alpha_i}$ such that $(\mathrm{d}u_{\alpha_i})(1) = e_i$ and $(\mathrm{d}u_{-\alpha_i})(1) = f_i$ for each $i \in I$. There exists an automorphism of G whose differential at the identity element coincides with the diagram automorphism ω of $\mathfrak{g}$ (cf. [Ja, II.1.13–15]). By abuse of notation, we will denote still by ω this automorphism of G and by $\langle \omega \rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(G)$ generated by the ω. Thus the automorphism ω of G permutes the root subgroups in such a way that

$$\omega(u_{\pm\alpha_i}(\xi)) = u_{\pm\alpha_{\omega(i)}}(\xi) \quad \text{for all } \xi \in \mathbb{C} \text{ and } i \in I.$$

Whenever there can be ambiguity, we will write $\mathrm{d}\omega$ for the automorphism of $\mathfrak{g}$.

Recall that the Weyl group $W \subset GL(\mathfrak{h}^*)$ may be identified with $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Each $w \in W^\omega$ lifts to an element of $N_G(T)$ fixed by $\omega \in \mathrm{Aut}(G)$ (cf. [Sp, 9.3]), which will be denoted by $\dot{w}$. We will also denote the induced action of ω on Λ by the same letter ω, and set $\Lambda^\omega = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$, $\Lambda^\omega_+ = \Lambda^\omega \cap \Lambda_+$. Note that, under the identification $\Lambda \simeq \mathfrak{h}^*_{\mathbb{Z}} \subset \mathfrak{h}^*$, this action of ω on Λ coincides with the restriction of $((\mathrm{d}\omega)^{-1})^* = ((\mathrm{d}\omega)^*)^{-1}$ to $\mathfrak{h}^*_{\mathbb{Z}}$.

By a $\langle\omega\rangle \ltimes G$-module M, we will always mean a finite-dimensional rational G-module that admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that

$$\omega \cdot (g\, m) = \omega(g)(\omega \cdot m) \quad \text{for all } g \in G,\ m \in M.$$

Regarding the semi-direct product $\langle\omega\rangle \ltimes G$ of $\langle\omega\rangle$ and G as a linear algebraic group, this is the same as a finite-dimensional rational $\langle\omega\rangle \ltimes G$-module. Likewise for $\langle\omega\rangle \ltimes B$- and $\langle\omega\rangle \ltimes T$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of Λ^ω with basis $e(\lambda)$, $\lambda \in \Lambda^\omega$. Let M be a $\langle\omega\rangle \ltimes T$-module, and let

$$M = \coprod_{\lambda \in \Lambda} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M \mid t\, m = \lambda(t) m \quad \text{for all } t \in T\}$$

be the weight space decomposition with respect to T. Now, following [FSS] and [FRS], we define the twining character $\mathrm{ch}^\omega(M)$ of M to be

$$\mathrm{ch}^\omega(M) = \sum_{\lambda \in \Lambda^\omega} \mathrm{Tr}(\omega|_{M_\lambda})\, e(\lambda) \in \mathbb{C}[\Lambda^\omega].$$

Remark 1.5.1. It easily follows that for each $t \in T$,

$$\mathrm{Tr}((\omega, t)\ ;\ M) = \sum_{\lambda \in \Lambda^\omega} \mathrm{Tr}(\omega|_{M_\lambda})\, \lambda(t) \in \mathbb{C}$$

since $\omega \cdot M_\lambda = M_{\omega \cdot \lambda}$ for $\lambda \in \Lambda$. Hence the twining character $\mathrm{ch}^\omega(M) \in \mathbb{C}[\Lambda^\omega]$ can be thought of as the trace function

$$T \ni t \mapsto \mathrm{Tr}((\omega, t)\ ;\ M) \in \mathbb{C}.$$

1.6. An example

Let $\lambda \in \Lambda^\omega_+$ and $L(\lambda)$ the simple rational G-module of highest weight λ. We can make $L(\lambda)$ into a $\langle\omega\rangle \ltimes G$-module as follows. The G-module ${}^\omega L(\lambda)$ obtained from $L(\lambda)$ by twisting the G-action by ω is isomorphic to $L(\lambda)$, since ω fixes λ. If τ_ω is the isomorphism from ${}^\omega L(\lambda)$ to $L(\lambda)$, define the $\langle\omega\rangle \ltimes G$-action on $L(\lambda)$ by

$$(\omega^r, g) \cdot v = \tau_\omega^{-r}(gv), \quad r \in \mathbb{Z},\ g \in G,\ v \in L(\lambda),$$

where gv on the right-hand side is computed with respect to the original G-action (cf. [FRS]). Note that a $\langle\omega\rangle \ltimes G$-module structure on $L(\lambda)$ such that ω fixes a highest weight vector v_λ of $L(\lambda)$ is unique since $L(\lambda)$ is a cyclic G-module generated by v_λ. Throughout the rest of this paper, by a $\langle\omega\rangle \ltimes G$-module $L(\lambda)$ we will always mean the one defined above, i.e., such that $\omega \cdot v_\lambda = v_\lambda$.

On the other hand, for each $i \in \widehat{I}$, we have by (1),

$$(P^*_\omega)^{-1}(\lambda)(N_i \widehat{h}_i) = \lambda(P_\omega^{-1}(N_i \widehat{h}_i)) = \lambda\left(\sum_{k=0}^{N_i - 1} h_{\omega^k(i)}\right) = N_i\, \lambda(h_i),$$

and hence $(P_\omega^*)^{-1}(\lambda)(\widehat{h}_i) = \lambda(h_i)$ (which also implies that $P_\omega^*(\widehat{\mathfrak{h}}_{\mathbb{Z}}^*) = (\mathfrak{h}_{\mathbb{Z}}^*)^0$). Thus $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ is dominant integral. If $\widehat{L}(\widehat{\lambda})$ is the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, it is shown in [FSS] and [FRS] that

$$\operatorname{ch}^\omega(L(\lambda)) = P_\omega^*\Big(\operatorname{ch} \widehat{L}(\widehat{\lambda})\Big), \tag{7}$$

where P_ω^* on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*] \tilde{\to} \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ defined by

$$P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu})) \quad \text{for } \widehat{\mu} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*.$$

Assume now that $J = I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$, $i \in \widehat{I}$, and let P_J be the standard parabolic subgroup of G associated to J. Let $\nu \in \Lambda^\omega$ with $\langle \nu, \alpha_i^\vee \rangle \ge 0$ (hence $\langle \nu, \alpha_j^\vee \rangle \ge 0$ for all $j \in J$). If $L_J(\nu)$ is the simple rational P_J-module of highest weight ν, then it remains simple as a rational module over the Levi factor L_J of P_J with the unipotent radical U_J of P_J acting trivially. We can make $L_J(\nu)$ into a $\langle\omega\rangle \ltimes P_J$-module in the same way as $L(\lambda)$ above.

Lemma 1.6.1. *With the notation and assumption as above, we have in* $\mathbb{C}[\Lambda^\omega]$,

$$\operatorname{ch}^\omega(L_J(\nu)) = \widehat{D}_i(e(\nu)).$$

Proof. Let $\mathfrak{g}_J$ be the reductive subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and $\{e_k, f_k \mid k \in J\}$, and $\widehat{\mathfrak{g}}_J \subset \widehat{\mathfrak{g}}$ the (reductive) orbit Lie algebra of $\mathfrak{g}_J$. If $\widehat{\nu} = (P_\omega^*)^{-1}(\nu) \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*$ and if $\widehat{L}_J(\widehat{\nu})$ is the simple $\widehat{\mathfrak{g}}_J$-module with highest weight $\widehat{\nu}$, then we have

$$\operatorname{ch}^\omega(L_J(\nu)) = P_\omega^*\Big(\operatorname{ch} \widehat{L}_J(\widehat{\nu})\Big)$$

since the proof of (7) in [FRS] goes through also for the reductive subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$. Moreover, because the (reductive) orbit Lie algebra $\widehat{\mathfrak{g}}_J$ is of type A_1 and $\widehat{\nu}(\widehat{h}_i) = \nu(h_i) \in \mathbb{Z}_{\ge 0}$, we deduce that

$$\begin{aligned} P_\omega^*(\operatorname{ch} \widehat{L}_J(\widehat{\nu})) &= P_\omega^*\Big(e(\widehat{\nu}) + e(\widehat{\nu} - \widehat{\alpha}_i) + \cdots + e(\widehat{r}_i(\widehat{\nu}))\Big) \\ &= P_\omega^*\Big(D_{\widehat{r}_i}(e(\widehat{\nu}))\Big) \\ &= \widehat{D}_i\big(P_\omega^*(e(\widehat{\nu}))\big) \quad \text{by (6)} \\ &= \widehat{D}_i(e(\nu)). \end{aligned}$$

This proves the lemma. ∎

2. The Demazure-Hansen desingularizations revisited

In this section we elaborate on the $\langle\omega\rangle \ltimes B$-equivariant Demazure-Hansen desingularizations of the $\langle\omega\rangle$-invariant Schubert varieties and the $\langle\omega\rangle \ltimes B$-equivariant sheaves on these varieties. For that we will desingularize the Schubert variety $X(w)$, $w \in W^\omega$, by a Bott-Samelson variety $X(w_{i_1}, \ldots, w_{i_r})$ along the reduced decomposition of w in the Coxeter system $(W^\omega, \{w_i \mid i \in \hat{I}\})$, and set up $\langle\omega\rangle \ltimes B$-equivariant Leray spectral sequences. We will also verify a $\langle\omega\rangle \ltimes B$-equivariant Serre duality. These will be our main tool to compute the twining character of the Demazure modules.

2.1. The Schubert and the Bott-Samelson varieties

For each $w \in W$, let $X(w)$ be the Zariski closure of an affine quotient $B\dot{w}B/B$ (called a Bruhat cell) in the flag variety G/B. If $y_1, \dots, y_n \in W$, let $X(y_1, \dots, y_n) = \{(g_1B, \dots, g_nB) \in (G/B)^n \mid g_{i-1}^{-1}g_i \in \overline{B\dot{y}_iB} \quad \text{for all } i\}$, called a Bott-Samelson variety.

If M is a B-module, regard M as a B^n-module via the n-th projection $B^n \to B$ and define an $\mathcal{O}_{X(y_1,\dots,y_n)}$-module $\mathcal{L}(M) = \mathcal{L}_{X(y_1,\dots,y_n)}(M)$ by setting on each open set V of $X(y_1, \dots, y_n)$

$$\begin{aligned}\Gamma(V, \mathcal{L}(M)) &= \mathbf{Var}(q^{-1}(V), M)^B \\ &= \{f \in \mathbf{Var}(q^{-1}(V), M) \mid f(xb) = b^{-1}f(x) \text{ in } M \text{ for each } x \in q^{-1}(V) \text{ and } b \in B\}\end{aligned}$$

where $q\colon \{(g_1, \dots, g_n) \in G^n \mid g_{i-1}^{-1}g_i \in \overline{B\dot{y}_iB} \quad \text{for all } i\} \to X(y_1, \dots, y_n)$ is the quotient.

If $J \subseteq I$, let P_J be the standard parabolic subgroup of G associated to J, and let z_J be the longest element of the Weyl group W_J of P_J. Then $X(z_J) = P_J/B$ is smooth, and hence also $X(z_{J_1}, \dots, z_{J_n})$ is smooth for subsets $J_1, \dots, J_n \subset I$ (cf. [Ja, II.13.5–6]). Due to Andersen, Ramanan-Ramanathan, and Seshadri (cf. [Ja, II.14.15.a)]), the Schubert variety $X(z_{J_1} \cdots z_{J_n})$ is normal. Put for simplicity $z_i = z_{J_i}$ for $1 \le i \le n$, $z = z_1 \cdots z_n$, and $X = X(z_1, \dots, z_n)$.

Lemma 2.1.1. *For each $i \in [0, n]$, let $X_i = X(z_1, \dots, z_i)$ and ${}_iX = X(z_{i+1}, \dots, z_n)$ with $X_0 = X(1) = {}_nX$. If M is a B-module with $H^j(X(z_i), \mathcal{L}_{X(z_i)}(H^0({}_iX, \mathcal{L}_{{}_iX}(M)))) = 0$ for all $i \in [1, n]$ and $j \ge 1$, then $H^j(X, \mathcal{L}_X(M)) = 0$ for all $j \ge 1$.*

Proof. Define $\pi_{ij} \in \mathbf{Var}(X_j, X_i)$ for each i, j with $0 \le i < j \le n$ by

$$(g_1B, \dots, g_iB, g_{i+1}B, \dots, g_jB) \mapsto (g_1B, \dots, g_iB).$$

Thus $\pi_{i_1,i_2} \circ \pi_{i_2,i_3} = \pi_{i_1,i_3} : X_{i_3} \to X_{i_1}$ if $0 \le i_1 < i_2 < i_3 \le n$. Put ${}_iM = H^0({}_iX, \mathcal{L}_{{}_iX}(M))$ for each $i \in [0, n]$. By the hypothesis and by [Ja, II.14.1(4)] we have

$$R^j(\pi_{i-1,i})_*\mathcal{L}_{X_i}({}_iM) \simeq \mathcal{L}_{X_{i-1}}(H^j(X(z_i), \mathcal{L}_{X(z_i)}({}_iM))) = 0 \quad \text{for all } j \ge 1,$$

hence the Leray spectral sequence $H^k(X_{i-1}, R^j(\pi_{i-1,i})_*\mathcal{L}_{X_i}({}_iM)) \Rightarrow H^{k+j}(X_i, \mathcal{L}_{X_i}({}_iM))$ degenerates to yield

$$H^j(X_{i-1}, (\pi_{i-1,i})_*\mathcal{L}_{X_i}({}_iM)) \simeq H^j(X_i, \mathcal{L}_{X_i}({}_iM)).$$

Since $\mathcal{L}_{X_i}({}_iM) \simeq (\pi_{in})_*\mathcal{L}_X(M)$ again by [Ja, II.14.1(4)], we obtain further

$$H^j(X_i, (\pi_{in})_*\mathcal{L}_X(M)) \simeq H^j(X_{i-1}, (\pi_{i-1,i})_*\mathcal{L}_{X_i}({}_iM)) \simeq H^j(X_{i-1}, (\pi_{i-1,n})_*\mathcal{L}_X(M)),$$

and hence

$$H^j(X, \mathcal{L}_X(M)) \simeq H^j(X_{n-1}, (\pi_{n-1,n})_*\mathcal{L}_X(M)) \simeq H^j(X_0, (\pi_{0n})_*\mathcal{L}_X(M)).$$

The last term vanishes for all $j \ge 1$ by the Grothendieck vanishing theorem since $X_0 = X(1)$ is a point. This proves the lemma. □

2.2. Cohomology vanishing

Keep the notation of §2.1, but assume that $\ell(z) = \ell(z_1) + \cdots + \ell(z_n)$. Let $\phi\colon X \to X(z)$ be the restriction to X of the n-th projection $(G/B)^n \to G/B$, which is a Demazure-Hansen desingularization of $X(z)$ (cf. [Ja, II.13.5(7),(8)]). As the Schubert variety $X(z)$ is normal, it follows from [Ja, II.14.5] that

$$\phi_*\mathcal{O}_X \simeq \mathcal{O}_{X(z)}. \tag{8}$$

Let M be a finite-dimensional B-module. We have from [Ja, Remark in I.5.17] that

$$\phi^*\mathcal{L}_{X(z)}(M) \simeq \mathcal{L}_X(M). \tag{9}$$

The sheaf $\mathcal{L}_{X(z)}(M)$ of $\mathcal{O}_{X(z)}$-modules is locally free of finite rank (cf. [Ja, I.5.16(2)]), and hence

$$\begin{aligned} R^\bullet\phi_*\mathcal{L}_X(M) &\simeq R^\bullet\phi_*(\phi^*\mathcal{L}_{X(z)}(M)) \quad \text{by (9)} \\ &\simeq (R^\bullet\phi_*\mathcal{O}_X) \otimes_{X(z)} \mathcal{L}_{X(z)}(M) \quad \text{by the projection formula [Ja, II.14.6(2)].} \end{aligned}$$

In particular, we obtain that

$$\begin{aligned} \phi_*\mathcal{L}_X(M) &\simeq (\phi_*\mathcal{O}_X) \otimes_{X(z)} \mathcal{L}_{X(z)}(M) \\ &\simeq \mathcal{L}_{X(z)}(M) \quad \text{by (8).} \end{aligned} \tag{10}$$

Taking the global sections of these yields

$$H^0(X, \mathcal{L}_X(M)) \simeq H^0(X(z), \mathcal{L}_{X(z)}(M)), \tag{11}$$

which is finite-dimensional over $\mathbb{C}$ by Serre's theorem.

For $\lambda \in \Lambda$, we let $\mathbb{C}_\lambda$ denote the one-dimensional B-module over $\mathbb{C}$ on which B acts via λ through the quotient $B \to T$. Note that if $\lambda \in \Lambda^\omega$, then by letting $\langle\omega\rangle$ act trivially on $\mathbb{C}_\lambda$ we have $\omega \cdot (b\,v) = b\,v = \omega(b)v = \omega(b)(\omega \cdot v)$ on $\mathbb{C}_\lambda$ for each $b \in B$ and $v \in \mathbb{C}_\lambda$. Now [Ja, II.14.15] generalizes as follows.

Theorem 2.2.1. *Let $z = z_1 \cdots z_n$ with $\ell(z) = \ell(z_1)+\cdots+\ell(z_n)$ and $X = X(z_1, \ldots, z_n)$.*

(i) *$R^j\phi_*\mathcal{O}_X = 0$ for all $j \geq 1$.*

(ii) *If $\lambda \in \Lambda_+$, then $H^j(X, \mathcal{L}_X(\mathbb{C}_\lambda)) = 0$ for all $j \geq 1$.*

(iii) *If $\mathcal{M}$ is a locally free $\mathcal{O}_{X(z)}$-module of finite rank, then we have $H^\bullet(X(z), \mathcal{M}) \simeq H^\bullet(X, \phi^*\mathcal{M})$.*

Proof. We will argue only for (ii). The rest follow just as in [Ja, II.14.15]. By Lemma 2.1.1 we have only to check that all $H^j(X(z_i), \mathcal{L}(H^0({}_iX, \mathcal{L}(\mathbb{C}_\lambda))))$ vanish for $i \in [1, n]$ and $j \geq 1$. We will suppress the obvious subscript from $\mathcal{L}$. Let $z_i = r_{i(1)} \cdots r_{i(a)}$ (resp. $z_{i+1} \cdots z_n = r_{i(a+1)} \cdots r_{i(b)}$) be a reduced expression of z_i (resp. $z_{i+1} \cdots z_n$) in the Coxeter system $(W, \{r_k \mid k \in I\})$. If $\phi_i : X(r_{i(1)}, \ldots, r_{i(a)}) \to X(z_i)$ is the desingularization, then

$$\begin{aligned} H^j(X(z_i), \mathcal{L}(H^0({}_iX, \mathcal{L}(\mathbb{C}_\lambda)))) &\simeq H^j(X(z_i), \mathcal{L}(H^0(X(z_{i+1} \cdots z_n), \mathcal{L}(\mathbb{C}_\lambda)))) \quad \text{by (11)} \\ &\simeq H^j(X(r_{i(1)}, \ldots, r_{i(a)}), \phi_i^*\mathcal{L}(H^0(X(z_{i+1} \cdots z_n), \mathcal{L}(\mathbb{C}_\lambda)))) \quad \text{by [Ja, II.14.15.c)]} \\ &\simeq H^j(X(r_{i(1)}, \ldots, r_{i(a)}), \mathcal{L}(H^0(X(z_{i+1} \cdots z_n), \mathcal{L}(\mathbb{C}_\lambda)))) \quad \text{by (9)} \\ &\simeq H^j(X(r_{i(1)}, \ldots, r_{i(a)}), \mathcal{L}(H^0(X(r_{i(a+1)}, \ldots, r_{i(b)}), \mathcal{L}(\mathbb{C}_\lambda)))) \quad \text{by (11) again} \\ &\simeq H^j(X(r_{i(1)}, \ldots, r_{i(a)}), (\pi_{ab})_*\mathcal{L}_{X(r_{i(1)}, \ldots, r_{i(b)})}(\mathbb{C}_\lambda)) \quad \text{by [Ja, II.14.1(4)],} \end{aligned}$$

the last term of which belongs to the setup of [Ja, II.14.15], and hence vanishes for all $j \geq 1$. □

Going back to the setup of §1, let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. Since $\ell(w) = \ell(w_{i_1}) + \cdots + \ell(w_{i_n})$ by Lemma 1.3.1, we obtain

Corollary 2.2.2. *If M is a finite-dimensional rational B-module, then*

$$H^\bullet(X(w), \mathcal{L}_{X(w)}(M)) \simeq H^\bullet(X(w_{i_1}, \dots, w_{i_n}), \mathcal{L}_{X(w_{i_1}, \dots, w_{i_n})}(M)).$$

2.3. $\langle\omega\rangle \ltimes B$-equivariant spectral sequences

If M is a $\langle\omega\rangle \ltimes B$-module, then $\mathcal{L}(M) = \mathcal{L}_{G/B}(M)$ carries a structure of $\langle\omega\rangle$-equivariant $\mathcal{O}_{G/B}$-module given by $\theta_\omega \in \mathbf{Mod}_{G/B}(\mathcal{L}(M), \omega_*\mathcal{L}(M))$ such that

$$f \mapsto \omega^{-1} \circ f \circ \omega, \quad f \in \mathcal{L}(M)(V) = \mathbf{Var}(q^{-1}(V), M)^B \tag{12}$$

for each open V of G/B, where $q\colon G \to G/B$ is the quotient. On the other hand, $\mathcal{L}(M)$ has the standard G-equivariant structure $\psi \in \mathbf{Mod}_{G \times G/B}(a^*\mathcal{L}(M), p^*\mathcal{L}(M))$, where $a\colon G \times G/B \to G/B$ is the G-action on G/B given by the multiplication from the left and $p\colon G \times G/B \to G/B$ is the projection. If $\psi' \in \mathbf{Mod}_{G/B}(\mathcal{L}(M), a_*p^*\mathcal{L}(M))$ is the adjoint of ψ, the two structures are intertwined by the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{L}(M) & \xrightarrow{\quad\psi'\quad} & & & a_*p^*\mathcal{L}(M) & = & a_*(\mathcal{O}_G \boxtimes_{\mathbb{C}} \mathcal{L}(M)) \\
\downarrow{\scriptstyle\theta_\omega} & & & & & & \downarrow{\scriptstyle a_*(\omega^\# \boxtimes_{\mathbb{C}} \theta_\omega)} \\
\omega_*\mathcal{L}(M) & \xrightarrow[\omega_*\psi']{} & \omega_*a_*p^*\mathcal{L}(M) & \underset{\sim}{—} & a_*(\omega \times \omega)_*p^*\mathcal{L}(M) & \underset{\sim}{—} & a_*(\omega_*\mathcal{O}_G \boxtimes_{\mathbb{C}} \omega_*\mathcal{L}(M)).
\end{array}$$

Thus, regarding $\langle\omega\rangle$ as a reduced algebraic group over $\mathbb{C}$ and forming a semi-direct product $\langle\omega\rangle \ltimes G$ of algebraic groups, $\mathcal{L}(M)$ comes equipped with a structure of $\langle\omega\rangle \ltimes G$-equivariant $\mathcal{O}_{G/B}$-module (cf. [MFK, 1.3]).

Now let $w \in W^\omega$ and let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. Put for simplicity $z_j = w_{i_j}$, $1 \le j \le n$. By Lemma 1.3.1 we have $\ell(w) = \ell(z_1) + \cdots + \ell(z_n)$, so that we may apply the results of §2.2. By our choice of a lift of each z_j in $N_G(T)^\omega$, the Bott-Samelson variety $X = X(z_1, \dots, z_n)$ admits a $\langle\omega\rangle$-action, and the desingularization $\phi\colon X(z_1, \dots, z_n) \to X(w)$ is $\langle\omega\rangle \ltimes B$-equivariant. Let M be a $\langle\omega\rangle \ltimes B$-module. Then the isomorphisms $\phi^*\mathcal{L}_{X(w)}(M) \simeq \mathcal{L}_X(M)$ from (9) and $\phi_*\mathcal{L}_X(M) \simeq \mathcal{L}_{X(w)}(M)$ from (10) are both $\langle\omega\rangle \ltimes B$-equivariant. By the $\langle\omega\rangle \ltimes B$-equivariance of the Leray spectral sequence induced by ϕ, the isomorphism of Corollary 2.2.2

$$H^\bullet(X(w), \mathcal{L}_{X(w)}(M)) \simeq H^\bullet(X(z_1, \dots, z_n), \mathcal{L}_{X(z_1, \dots, z_n)}(M)) \tag{13}$$

is $\langle\omega\rangle \ltimes B$-equivariant.

If $\pi_1 : X(z_1, \dots, z_n) \to X(z_1)$ is the projection onto the first factor, since π_1 is $\langle\omega\rangle \ltimes B$-equivariant, the Leray spectral sequence

$$H^i(X(z_1), R^j\pi_{1*}\mathcal{L}_{X(z_1, \dots, z_n)}(M)) \Rightarrow H^{i+j}(X(z_1, \dots, z_n), \mathcal{L}_{X(z_1, \dots, z_n)}(M)) \tag{14}$$

is $\langle\omega\rangle \ltimes B$-equivariant. Also, if we make $\mathcal{L}_{X(z_1)}(H^0(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(M)))$ into a $\langle\omega\rangle \ltimes B$-equivariant sheaf as in (12) by using the $\langle\omega\rangle \ltimes B$-module structure on $H^0(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(M))$, then the isomorphism (cf. [Ja, II.14.1(4)])

$$\mathcal{L}_{X(z_1)}(H^0(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(M))) \simeq \pi_{1*}\mathcal{L}_{X(z_1,\dots,z_n)}(M) \tag{15}$$

is $\langle\omega\rangle \ltimes B$-equivariant via the correspondence

$$f \mapsto \tilde{f} \quad \text{with} \quad \tilde{f}(x,y) = f(x)(y), \tag{16}$$

where $f \in \mathbf{Var}(q^{-1}(V), H^0(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(M)))^B$ with $q\colon P_{I_{i_1}}(\mathbb{C}) \to X(z_1)$ the quotient and V an open of $X(z_1)$, and $\tilde{f} \in \mathbf{Var}(q^{-1}(V)\times_{\mathbb{C}} V(z_2,\dots,z_n), M)^{B\times_{\mathbb{C}} B^{r-1}}$ with $V(z_2,\dots,z_n) = \{(g_2,\dots,g_n) \in G^{n-1} \mid g_{i-1}^{-1}g_i \in B\dot{z}_iB \quad \text{for all } i\}$. Taking the derived functors, we obtain a $\langle\omega\rangle \ltimes B$-equivariant isomorphism

$$R^\bullet\pi_{1*}\mathcal{L}_{X(z_1,\dots,z_n)}(M) \simeq \mathcal{L}_{X(z_1)}(H^\bullet(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(M))),$$

and hence a $\langle\omega\rangle \ltimes B$-equivariant spectral sequence

$$H^i(X(z_1),\mathcal{L}(H^j(X(z_2,\dots,z_n),\mathcal{L}(M)))) \Rightarrow H^{i+j}(X(z_1,\dots,z_n),\mathcal{L}(M)). \tag{17}$$

2.4. $\langle\omega\rangle \ltimes B$-equivaraint Serre duality

Now let $P = P_J$ be a standard parabolic subgroup of G with J an ω-invariant subset of I. Let $\Omega^1_{P/B}$ be the $\mathcal{O}_{P/B}$-module of the 1-differentials over $\mathbb{C}$ and $\Omega^n_{P/B} = \bigwedge^n_{P/B}\Omega^1_{P/B}$ with $n = \dim_{\mathbb{C}}(P/B)$. The $\langle\omega\rangle \ltimes P$-action on P/B makes $\Omega^n_{P/B}$ into a $\langle\omega\rangle \ltimes P$-equivariant, a fortiori, $\langle\omega\rangle \ltimes B$-equivariant $\mathcal{O}_{P/B}$-module. If $\mathcal{M}$ is a $\langle\omega\rangle \ltimes B$-equivariant $\mathcal{O}_{P/B}$-module that is locally free of finite rank over $\mathcal{O}_{P/B}$, we will need a $\langle\omega\rangle \ltimes B$-equivariant Serre duality

$$H^i(P/B,\mathcal{M}^\vee \otimes_{P/B} \Omega^n_{P/B}) \simeq H^{n-i}(P/B,\mathcal{M})^* \quad \text{for all } i \in [0,n], \tag{18}$$

where $\mathcal{M}^\vee = \mathcal{M}od_{P/B}(\mathcal{M},\mathcal{O}_{P/B})$ and $H^{n-i}(P/B,\mathcal{M})^*$ is the dual $\langle\omega\rangle \ltimes B$-module of $H^{n-i}(P/B,\mathcal{M})$.

Put for simplicity $X = P/B$. The plain (nonequivariant) Serre duality asserts that the Yoneda-Cartier pairing (cf. [AK, IV, Th. (1.1)], [Iv, I.8])

$$\mathrm{Ext}^i_X(\mathcal{M},\Omega^n_X) \times H^{n-i}(X,\mathcal{M}) \to H^n(X,\Omega^n_X)$$

is perfect. The standard argument verifies the pairing to be $\langle\omega\rangle \ltimes B$-equivariant, which yields a $\langle\omega\rangle \ltimes B$-equivariant isomorphism $H^{n-i}(X,\mathcal{M}) \simeq H^i(X,\mathcal{M}^\vee \otimes_X \Omega^n_X)^* \otimes_{\mathbb{C}} H^n(X,\Omega^n_X)$. Thus the $\langle\omega\rangle \ltimes B$-equivariance of (18) will be a consequence of the triviality of the $\langle\omega\rangle \ltimes B$-action on $H^n(X,\Omega^n_X)$. Note that the ω-action on Ω^n_X is not trivial (cf. (19) and (27) below).

To see the triviality of the action on $H^n(X,\Omega^n_X)$, take a $\langle\omega\rangle \ltimes P$-module V and a $\langle\omega\rangle \ltimes P$-equivariant closed immersion $\imath : X \to \mathbb{P}(V)$; for example, a simple rational P-module of a sufficiently dominant ω-fixed highest weight will do. Set $\mathbb{P} = \mathbb{P}(V)$, $v = \dim_{\mathbb{C}}\mathbb{P}$, and $\Omega^v_{\mathbb{P}} = \bigwedge^v_{\mathbb{P}}\Omega^1_{\mathbb{P}}$. Since $\mathbf{Var}(\mathbb{P},\mathbb{P})^\times \simeq PGL(V)$ (cf. [Ha, II, Example

7.1]) and since $H^v(\mathbb{P}, \Omega^v_{\mathbb{P}})$ is one-dimensional, $PGL(V)$ acts trivially on $H^v(\mathbb{P}, \Omega^v_{\mathbb{P}})$, so therefore does $\langle\omega\rangle \ltimes P$. On the other hand, the isomorphism of $\mathcal{O}_X$-modules (cf. [AK, I, Th. (4.6)])

$$\Omega^n_X \simeq \imath^* \mathcal{E}xt^r_{\mathbb{P}}(\imath_* \mathcal{O}_X, \Omega^v_{\mathbb{P}}) \quad \text{with } r = v - n$$

is $\langle\omega\rangle \ltimes P$-equivariant, and the $\mathbb{C}$-linear isomorphism (cf. [Ha, III, Lemma 7.4])

$$\varepsilon\colon \mathbf{Mod}_X(\Omega^n_X, \imath^* \mathcal{E}xt^r_{\mathbb{P}}(\imath_* \mathcal{O}_X, \Omega^v_{\mathbb{P}})) \to \operatorname{Ext}^r_{\mathbb{P}}(\imath_* \Omega^n_X, \Omega^v_{\mathbb{P}})$$

is $\langle\omega\rangle \ltimes P$-equivariant. In the commutative diagram

$$\begin{array}{ccc}
\operatorname{Ext}^r_{\mathbb{P}}(\imath_*\Omega^n_X, \Omega^v_{\mathbb{P}}) \times H^n(\mathbb{P}, \imath_*\Omega^n_X) & \longrightarrow & H^v(\mathbb{P}, \Omega^v_{\mathbb{P}}) \\
\uparrow {\scriptstyle \varepsilon \times H^n(\mathbb{P}, \imath_*\Omega^n_X)} & & \uparrow \\
\mathbf{Mod}_X(\Omega^n_X, \imath^* \mathcal{E}xt^r_{\mathbb{P}}(\imath_* \mathcal{O}_X, \Omega^v_{\mathbb{P}})) \times H^n(\mathbb{P}, \imath_*\Omega^n_X) & & \\
\uparrow \wr & & \\
\mathbf{Mod}_X(\Omega^n_X, \Omega^n_X) \times H^n(\mathbb{P}, \imath_*\Omega^n_X) & \xleftarrow[\mathrm{id}_{\Omega^n_X} \times H^n(\mathbb{P}, \imath_*\Omega^n_X)]{} & H^n(\mathbb{P}, \imath_*\Omega^n_X),
\end{array}$$

we have $\dim_{\mathbb{C}} H^v(\mathbb{P}, \Omega^v_{\mathbb{P}}) = 1 = \dim_{\mathbb{C}} H^n(\mathbb{P}, \imath_*\Omega^n_X)$ and the top horizontal map is a perfect pairing, hence the right vertical map is bijective. Since $\mathrm{id}_{\Omega^n_X}$ is obviously $\langle\omega\rangle \ltimes P$-equivariant, its image in $\operatorname{Ext}^r_{\mathbb{P}}(\imath_*\Omega^n_X, \Omega^v_{\mathbb{P}})$ is fixed under $\langle\omega\rangle \ltimes P$, and hence the right vertical isomorphism is $\langle\omega\rangle \ltimes P$-equivariant. It follows that $H^n(X, \Omega^n_X) \simeq H^n(\mathbb{P}, \imath_*\Omega^n_X)$ must be a trivial $\langle\omega\rangle \ltimes P$-module, as desired.

Note finally that we have an isomorphism of $\langle\omega\rangle \ltimes P$-equivariant $\mathcal{O}_{P/B}$-modules

$$\Omega^1_{P/B} \simeq \mathcal{L}_{P/B}((\operatorname{Lie}(P)/\operatorname{Lie}(B))^*), \tag{19}$$

and hence the $\langle\omega\rangle \ltimes B$-equivariant Serre duality (18) reads, for each $i \in [0, n]$, as

$$H^i(P/B, \mathcal{M}^\vee \otimes_{P/B} \mathcal{L}_{P/B}(\textstyle\bigwedge^n_{\mathbb{C}}(\operatorname{Lie}(P)/\operatorname{Lie}(B))^*)) \simeq H^{n-i}(P/B, \mathcal{M})^*. \tag{20}$$

3. Twining character formula for Demazure modules

Resume the setup of §2. Fix $w \in W^\omega$ and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$. For a $\langle\omega\rangle \ltimes B$-module M, the ω-Euler characteristic $\chi^\omega_w(M)$ is defined to be

$$\chi^\omega_w(M) = \sum_{j \geq 0} (-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) \in \mathbb{C}[\Lambda^\omega].$$

Let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. We will show in this section that

$$\chi^\omega_w(M) = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}(\operatorname{ch}^\omega(M)),$$

where $\widehat{D}_j$ for $j = i_1, \ldots, i_n$ is the ω-Demazure operator defined in §1.4. In particular, we will obtain a twining character formula of the Demazure module $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for $\lambda \in \Lambda^\omega_+$, where $\mathbb{C}_\lambda$ is the one-dimensional $\langle\omega\rangle \ltimes B$-module on which B acts by the weight λ through the quotient $B \to T$ and $\langle\omega\rangle$ trivially.

3.1. Formula for the ω-Euler characteristics
Set $\widehat{D}_w = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}$. Then we are to show

$$\chi_w^\omega(M) = \widehat{D}_w(\mathrm{ch}^\omega(M)). \tag{21}$$

Let us first make some reductions. Since both sides of (21) are additive in M, replacing M by $M^U = \{m \in M \mid u\,m = m \quad \text{for all } u \in U\}$, we may assume that the unipotent radical U of the Borel subgroup $B = T \ltimes U$ acts trivially on M. Consider the T-weight space decomposition $M = \coprod_{\nu \in \Lambda} M_\nu$. Let us denote by $\Lambda/\langle\omega\rangle$ a complete set of representatives of the $\langle\omega\rangle$-orbits in Λ, and set for each $\mu \in \Lambda/\langle\omega\rangle$,

$$M^{(\mu)} = \coprod_{\nu \in \langle\omega\rangle \cdot \mu} M_\nu.$$

Then we have a direct sum decomposition of M

$$M = \coprod_{\mu \in \Lambda/\langle\omega\rangle} M^{(\mu)}.$$

Note that each $M^{(\mu)}$ is a $\langle\omega\rangle \ltimes B$-submodule of M, and that both sides of (21) vanish on $M^{(\mu)}$ unless $\mu \in \Lambda^\omega$ (in which case $M^{(\mu)} = M_\mu$). Moreover, since $\omega^N = 1$, the action of ω on M_μ for $\mu \in \Lambda^\omega$ is semi-simple. Hence, by the additivity in M of both sides of (21), we may assume that M is one-dimensional of weight $\mu \in \Lambda^\omega$ on which ω is acting by a scalar ζ^k for a primitive N-th root of unity ζ in $\mathbb{C}$ and $k \in \mathbb{Z}$. We will denote such M by $\mathbb{C}_{\mu,k}$. Thus we are reduced to showing that

$$\chi_w^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_w(\mathrm{ch}^\omega(\mathbb{C}_{\mu,k})), \tag{22}$$

where $\mathrm{ch}^\omega(\mathbb{C}_{\mu,k}) = \zeta^k\, e(\mu)$.

Put for simplicity $z_j = w_{i_j}$, $1 \le j \le n$. From §2.3 we have an isomorphism (13) of $\langle\omega\rangle \ltimes B$-modules

$$H^\bullet(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\mu,k})) \simeq H^\bullet(X(z_1, \dots, z_n), \mathcal{L}_{X(z_1,\dots,z_n)}(\mathbb{C}_{\mu,k})), \tag{23}$$

and for each $s \in [1, n-1]$, a $\langle\omega\rangle \ltimes B$-equivariant spectral sequence (17)

$$H^i(X(z_s), \mathcal{L}(H^j(X(z_{s+1}, \dots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})))) \Rightarrow H^{i+j}(X(z_s, \dots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})). \tag{24}$$

It follows that

$$\begin{aligned}
\chi_w^\omega(\mathbb{C}_{\mu,k}) &= \sum_{j \ge 0} (-1)^j \,\mathrm{ch}^\omega(H^j(X(z_1, \dots, z_n), \mathcal{L}_{X(z_1,\dots,z_n)}(\mathbb{C}_{\mu,k}))) \quad \text{by (23)} \\
&= \sum_{j \ge 0} (-1)^j \left(\sum_{i \ge 0} (-1)^i \,\mathrm{ch}^\omega(H^i(X(z_1), \mathcal{L}(H^j(X(z_2, \dots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k}))))) \right) \quad \text{by (24)} \\
&= \sum_{j \ge 0} (-1)^j \,\chi_{z_1}^\omega(H^j(X(z_2, \dots, z_n), \mathcal{L}_{X(z_2,\dots,z_n)}(\mathbb{C}_{\mu,k}))).
\end{aligned} \tag{25}$$

We will prove (22) by induction on n. Now assume that $n = 1$ and that $w = w_i$ for some $i \in \widehat{I}$. So put $J = I_i$ and let $P = P_J$ be the standard parabolic subgroup of G associated to J. We are to show

$$\chi_{w_i}^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_i(\zeta^k e(\mu)). \tag{26}$$

Assume first that $\langle\mu, \alpha_i^\vee\rangle \ge 0$ (and hence that $\langle\mu, \alpha_k^\vee\rangle \ge 0$ for all $k \in J$). Let $L_J(\mu)$ be the simple rational P_J-module of highest weight μ admitting a $\langle\omega\rangle$-action as in §1.6, and let ζ^k be the one-dimensional trivial P_J-module with ω acting by the scalar ζ^k.

Lemma 3.1.1. *Let the notation and assumption be as above. Then we have the following isomorphism of $\langle\omega\rangle \ltimes P_J$-modules.*

$$H^0(P_J/B, \mathcal{L}_{P_J/B}(\mathbb{C}_{\mu,k})) \simeq L_J(\mu) \otimes_{\mathbb{C}} \zeta^k.$$

Proof. The left-hand side realizes a simple rational P_J-module of highest weight μ (cf. [Ja, I.6.11 and II.4.6]). Since ω acts on its (nonzero) highest weight vector by the scalar ζ^k (cf. [Ja, II.2.6]), the assertion follows. □

Now we deduce that

$$\begin{aligned}\chi^{\omega}_{w_i}(\mathbb{C}_{\mu,k}) &= \mathrm{ch}^{\omega}(H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))) \quad \text{by Kempf's vanishing theorem [Ja, II.4.5]}\\ &= \mathrm{ch}^{\omega}(L_J(\mu) \otimes_{\mathbb{C}} \zeta^k) \quad \text{by Lemma 3.1.1}\\ &= \zeta^k \mathrm{ch}^{\omega}(L_J(\mu))\\ &= \zeta^k\, \widehat{D}_i(e(\mu)) \quad \text{by Lemma 1.6.1}\\ &= \widehat{D}_i(\zeta^k e(\mu)).\end{aligned}$$

If $\langle \mu, \alpha_i^\vee\rangle = -1$ (and hence $\langle \mu, \alpha_k^\vee\rangle = -1$ for all $k \in J$), then both sides of (26) vanish (cf. [Ja, II.5.5]).

Assume finally that $\langle \mu, \alpha_i^\vee\rangle \le -2$ (and hence that $\langle \mu, \alpha_k^\vee\rangle \le -2$ for all $k \in J$). Set $\rho_J = \frac{1}{2}\sum_{\alpha\in\Delta_J^+}\alpha$ with $\Delta_J^+ = \Delta_+ \cap \sum_{k\in J}\mathbb{Z}\alpha_k$ the positive root system of P_J. By direct checking (see the proof of [N3, Prop. 3.2.2]), using the $\langle\omega\rangle \ltimes T$-module isomorphism $(\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \bigoplus_{\alpha\in\Delta_J^+}\mathbb{C}f_\alpha$, we see that as $\langle\omega\rangle \ltimes B$-modules,

$$\textstyle\bigwedge_{\mathbb{C}}^{\ell(w_i)}(\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \mathbb{C}_{-2\rho_J,0} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}, \tag{27}$$

where $\ell(w_i) = \dim_{\mathbb{C}}(P/B)$ and $(-1)^{\ell(w_i)-1}$ is the one-dimensional $\langle\omega\rangle \ltimes B$-module with B acting trivially and ω by the scalar $(-1)^{\ell(w_i)-1}$. Then the $\langle\omega\rangle \ltimes B$-equivariant Serre duality (20) reads

$$\begin{aligned}H^j(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))^* &\simeq H^{\ell(w_i)-j}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}))\\ &\simeq \begin{cases} H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k})) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} & \text{if } j = \ell(w_i),\\ 0 & \text{otherwise (by Kempf).}\end{cases}\end{aligned} \tag{28}$$

Lemma 3.1.2. *Let J be an ω-invariant subset of I, w_J the longest element of the Weyl group W_J of P_J, and let $\nu \in \Lambda^\omega$ be such that $\langle \nu, \alpha_k^\vee\rangle \ge 0$ for all $k \in J$. Then we have the following isomorphism of $\langle\omega\rangle \ltimes P_J$-modules.*

$$L_J(\nu)^* \simeq L_J(-w_J(\nu)).$$

Proof. Note that a (nonzero) highest weight vector v_+^* of the dual module $L_J(\nu)^*$ is the dual element of a lowest weight vector $\dot{w}_J\, v_+$ of $L_J(\nu)$, with v_+ a (nonzero) highest weight vector of $L_J(\nu)$. Since $w_J \in W^\omega$ is fixed by ω (cf. Remark 1.3.2), so is $\dot{w}_J\, v_+$, and hence also $v_+^* \in L_J(\nu)^*$. This proves the lemma. □

The isomorphism (28) together with Lemmas 3.1.1 and 3.1.2 implies that, as $\langle\omega\rangle \ltimes B$-modules,

$$
\begin{aligned}
H^{\ell(w_i)}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k})) &\simeq \Big(L_J(-\mu-2\rho_J)^* \otimes_{\mathbb{C}} \zeta^k\Big) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} \\
&\simeq L_J(w_i(\mu+2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}. \qquad (29)
\end{aligned}
$$

Then, setting $\widehat{\mu} = (P_\omega^*)^{-1}(\mu)$,

$$
\begin{aligned}
\chi_{w_i}^\omega(\mathbb{C}_{\mu,k}) &= (-1)^{\ell(w_i)} \operatorname{ch}^\omega(L_J(w_i(\mu+2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}) \quad \text{by (29)} \\
&= -\zeta^k \operatorname{ch}^\omega(L_J(w_i(\mu+2\rho_J))) \\
&= -\zeta^k\, \widehat{D}_i(e(w_i(\mu+2\rho_J))) \quad \text{by Lemma 1.6.1} \\
&= -\zeta^k \Big(P_\omega^* \circ D_{\widehat{r}_i} \circ (P_\omega^*)^{-1}\Big)(e(w_i(\mu+2\rho_J))) \quad \text{by (6)} \\
&= -\zeta^k \Big(P_\omega^* \circ D_{\widehat{r}_i}\Big)(e(\widehat{r}_i(\widehat{\mu}+\widehat{\alpha}_i))) \quad \text{since } (P_\omega^*)^{-1}(2\rho_J) = \widehat{\alpha}_i \\
&= -\zeta^k\, P_\omega^*(-D_{\widehat{r}_i}(e(\widehat{\mu}))) \\
&= \zeta^k \Big(\widehat{D}_i \circ P_\omega^*\Big)(e(\widehat{\mu})) \quad \text{by (6)} \\
&= \zeta^k\, \widehat{D}_i(e(\mu)) \\
&= \widehat{D}_i(\zeta^k e(\mu)).
\end{aligned}
$$

Thus in all cases (26) holds, and hence

$$
\chi_{w_i}^\omega(M) = \widehat{D}_i(\operatorname{ch}^\omega(M)) \qquad (30)
$$

holds for any $\langle\omega\rangle \ltimes B$-module M.

Now let us return to the original setup, and let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. Then we get that

$$
\begin{aligned}
\chi_w^\omega(\mathbb{C}_{\mu,k}) &= \sum_{j\geq 0} (-1)^j\, \chi_{w_{i_1}}^\omega(H^j(X(w_{i_2}, \ldots, w_{i_n}), \mathcal{L}_{X(w_{i_2},\ldots,w_{i_n})}(\mathbb{C}_{\mu,k}))) \quad \text{by (25)} \\
&= \sum_{j\geq 0} (-1)^j\, \widehat{D}_{i_1}\Big(\operatorname{ch}^\omega(H^j(X(w_{i_2}, \ldots, w_{i_n}), \mathcal{L}_{X(w_{i_2},\ldots,w_{i_n})}(\mathbb{C}_{\mu,k})))\Big) \quad \text{by (30)} \\
&= \widehat{D}_{i_1}\left(\sum_{j\geq 0} (-1)^j \operatorname{ch}^\omega(H_j(X(w_{i_2}, \ldots, w_{i_n}), \mathcal{L}_{X(w_{i_2},\ldots,w_{i_n})}(\mathbb{C}_{\mu,k})))\right) \\
&= \widehat{D}_{i_1}\Big(\chi_{w_{i_2}\cdots w_{i_n}}^\omega(\mathbb{C}_{\mu,k})\Big).
\end{aligned}
$$

Here, since $\widehat{\ell}(w_{i_2} \cdots w_{i_n}) = n-1$, we have by the induction hypothesis that

$$
\chi_{w_{i_2}\cdots w_{i_n}}^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_{i_2} \cdots \widehat{D}_{i_n}(\operatorname{ch}^\omega(\mathbb{C}_{\mu,k})).
$$

Therefore, we finally obtain that

$$\chi_w^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}(\mathrm{ch}^\omega(\mathbb{C}_{\mu,k})),$$

proving (22) and hence (21).

If $\lambda \in \Lambda_+^\omega$, then for any Schubert variety $X(w)$,

$$H^j(X(w), \mathcal{L}_{X(w)}(\lambda)) = 0 \quad \text{for all } j \geq 1$$

by the Demazure vanishing theorem [Ja, II.14.15] of Andersen, Mehta-Ramanathan, Ramanan-Ramanathan, and Seshadri (cf. also [Kan]). Hence we have obtained

$$\chi_w^\omega(M) = \sum_{j\geq 0}(-1)^j\,\mathrm{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) = \widehat{D}_w(\mathrm{ch}^\omega(M)) \in \mathbb{C}[\Lambda^\omega]$$

for a finite-dimensional rational $\langle\omega\rangle \ltimes B$-module M and $w \in W^\omega$, and in particular,

$$\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_w(e(\lambda))$$

for $\lambda \in \Lambda_+^\omega$. Namely, we have proved Theorem 0.1 in the introduction.

Theorem 0.1 reveals that there exists a striking relation between the ω-Euler characteristic $\chi_w^\omega(\mathbb{C}_\lambda) \in \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ for $\mathfrak{g}$ and the ordinary Euler characteristic for the orbit Lie algebra $\widehat{\mathfrak{g}}$. To state the relation, we need some notation. Recall from Remark 1.2.1 that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the dual Lie algebra ${}^t\big(({}^t\mathfrak{g})^{{}^t\omega}\big)$ of the (semi-simple) fixed point subalgebra $({}^t\mathfrak{g})^{{}^t\omega} = \{x \in {}^t\mathfrak{g} \mid ({}^t\omega)(x) = x\}$ of ${}^t\mathfrak{g}$ by ${}^t\omega \in \mathrm{Aut}({}^t\mathfrak{g})$. Let $\widehat{G}$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $\widehat{T}$ and Borel subgroup $\widehat{B} \supset \widehat{T}$ such that $\mathrm{Lie}(\widehat{G}) = \widehat{\mathfrak{g}}$, $\mathrm{Lie}(\widehat{T}) = \widehat{\mathfrak{h}}$, and $\mathrm{Lie}(\widehat{B}) = \widehat{\mathfrak{b}}$. For $\widehat{w} \in \widehat{W} \simeq N_{\widehat{G}}(\widehat{T})/\widehat{T}$, we take a right coset representative $\dot{\widehat{w}} \in N_{\widehat{G}}(\widehat{T})$ of $\widehat{w}$, and define the Schubert variety $\widehat{X}(\widehat{w})$ over $\mathbb{C}$ by

$$\widehat{X}(\widehat{w}) = \overline{\widehat{B}\dot{\widehat{w}}\widehat{B}/\widehat{B}} = \overline{\widehat{B}\dot{\widehat{w}}\widehat{B}}/\widehat{B} \subset \widehat{G}/\widehat{B}.$$

For each $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, we denote by $\mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})$ the (locally free) $\widehat{B}$-equivariant sheaf of $\mathcal{O}_{\widehat{X}(\widehat{w})}$-modules associated to the one-dimensional $\widehat{B}$-module $\mathbb{C}_{\widehat{\lambda}}$ on which $\widehat{B}$ acts by the weight $\widehat{\lambda}$ through the quotient $\widehat{B} \to \widehat{T}$.

Now we are ready to state the following

Corollary 3.1.3. *Let $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$ and $w \in W^\omega$. We set $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$ and $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$. Then we have in the algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$,*

$$\begin{aligned}\chi_w^\omega(\mathbb{C}_\lambda) &= \sum_{j\geq 0}(-1)^j\,\mathrm{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))\\ &= P_\omega^*\left(\sum_{j\geq 0}(-1)^j\,\mathrm{ch}\,H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}}))\right),\end{aligned}$$

where $\mathrm{ch}\,H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) \in \mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ for $j \in \mathbb{Z}_{\geq 0}$ is the ordinary character of the j-th cohomology group $H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}}))$ of $\widehat{X}(\widehat{w})$.

Proof. Let $w = w_{i_1} w_{i_2} \cdots w_{i_n}$ be a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. Then $\widehat{w} = \Theta^{-1}(w) = \widehat{r}_{i_1} \widehat{r}_{i_2} \cdots \widehat{r}_{i_n}$ is a reduced expression of $\widehat{w} \in \widehat{W}$ in the Coxeter system $(\widehat{W}, \{\widehat{r}_i \mid i \in \widehat{I}\})$. Hence, by the ordinary Demazure character formula [Ja, II.14.18] for the orbit Lie algebra $\widehat{\mathfrak{g}}$, we have in the algebra $\mathbb{C}[\widehat{\mathfrak{h}}^*_{\mathbb{Z}}]$,

$$\sum_{j \geq 0} (-1)^j \operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) = D_{\widehat{r}_{i_1}} D_{\widehat{r}_{i_2}} \cdots D_{\widehat{r}_{i_n}}(e(\widehat{\lambda})). \tag{31}$$

By applying P^*_ω to both sides of the equality (31), we obtain in the algebra $\mathbb{C}[(\mathfrak{h}^*_{\mathbb{Z}})^0]$,

$$\begin{aligned} P^*_\omega \left(\sum_{j \geq 0} (-1)^j \operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) \right) &= P^*_\omega \left(D_{\widehat{r}_{i_1}} D_{\widehat{r}_{i_2}} \cdots D_{\widehat{r}_{i_n}}(e(\widehat{\lambda})) \right) \quad \text{by (31)} \\ &= \widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_n}(P^*_\omega(e(\widehat{\lambda}))) \quad \text{by (6)} \\ &= \widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_n}(e(\lambda)) \\ &= \sum_{j \geq 0} (-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))), \end{aligned}$$

where the last equality is by Theorem 0.1. This proves the corollary. □

3.2. Joseph's modules

Let us finally return to Joseph's module $J_w(\lambda)$ with $w \in W^\omega$ and $\lambda \in \Lambda^\omega_+$. Thus let v^*_λ be a (nonzero) lowest weight vector of the dual module $L(\lambda)^*$ (which is the dual element of a (nonzero) highest weight vector v_λ of $L(\lambda)$), and let $\dot{w} \in N_G(T)^\omega$ be a representative of $w \in W^\omega$. Since v^*_λ is fixed by ω, so is $\dot{w}\, v^*_\lambda$. Joseph's module $J_w(\lambda)$ of highest weight $-w(\lambda)$ in $L(\lambda)^*$ is defined to be

$$J_w(\lambda) = \mathfrak{U}(\mathfrak{b})\, (\dot{w}\, v^*_\lambda) \subset L(\lambda)^*,$$

where $\mathfrak{U}(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b} = \operatorname{Lie}(B)$. Note that, since $\omega \cdot (\dot{w}\, v^*_\lambda) = \dot{w}\, v^*_\lambda$, Joseph's module $J_w(\lambda)$ is a $\langle\omega\rangle \ltimes B$-submodule of $L(\lambda)^*$. Moreover, since $\dot{w}_0\, v^*_\lambda$ is a (nonzero) highest weight vector of $L(\lambda)^*$ fixed by ω, there is an isomorphism of $\langle\omega\rangle \ltimes G$-modules

$$L(\lambda)^* \simeq L(-w_0(\lambda)), \tag{32}$$

which enables us to regard $J_w(\lambda)$ as a $\langle\omega\rangle \ltimes B$-submodule of $L(-w_0(\lambda))$. Then we obtain a short exact sequence of $\langle\omega\rangle \ltimes B$-modules

$$0 \leftarrow J_w(\lambda)^* \leftarrow L(-w_0(\lambda))^* \leftarrow J_w(\lambda)^\perp \leftarrow 0,$$

with $J_w(\lambda)^\perp = \{\phi \in L(-w_0(\lambda))^* \mid \phi(J_w(\lambda)) = 0\}$. On the other hand, Lemma 3.1.1 for the case $J = I$ combined with (32) yields an isomorphism of $\langle\omega\rangle \ltimes G$-modules

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \simeq L(-w_0(\lambda))^*.$$

Since the restriction map

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \to H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$$

is a $\langle \omega \rangle \ltimes B$-equivariant surjection by [Ja, II.14.19], we obtain from [Ja, II.14.19(2)] a commutative diagram of short exact sequences of $\langle \omega \rangle \ltimes B$-modules

$$\begin{array}{ccccccccc}
0 & \longleftarrow & J_w(\lambda)^* & \longleftarrow & L(-w_0(\lambda))^* & \longleftarrow & J_w(\lambda)^{\perp} & \longleftarrow & 0 \\
 & & & & \wr\big| & & \big\| & & \\
0 & \longleftarrow & H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)) & \longleftarrow & H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) & \longleftarrow & J_w(\lambda)^{\perp} & \longleftarrow & 0,
\end{array}$$

and hence an isomorphism of $\langle \omega \rangle \ltimes B$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)),$$

or equivalently

$$J_w(\lambda) \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*. \tag{33}$$

Here we employ an elementary lemma from linear algebra.

Lemma 3.2.1. *Let $S \in M(m, \mathbb{C})$ be an $m \times m$ complex matrix such that $\mathrm{Tr}(S) \in \mathbb{R}$. Suppose that $S^k = I_m$ for some $k \in \mathbb{Z}_{\geq 1}$, where I_m is the identity matrix. Then we have that $\mathrm{Tr}(S^{-1}) = \mathrm{Tr}(S)$.*

We now define a $\mathbb{C}$-linear conjugation $^{-} : \mathbb{C}[\Lambda^\omega] \to \mathbb{C}[\Lambda^\omega]$ by

$$\overline{\sum_{\mu \in \Lambda^\omega} a_\mu \, e(\mu)} = \sum_{\mu \in \Lambda^\omega} a_\mu \, e(-\mu) \quad \text{with } a_\mu \in \mathbb{C} \text{ for } \mu \in \Lambda^\omega.$$

Theorem 3.2.2. *Let $\lambda \in \Lambda_+^\omega$ and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))}.$$

Proof. By (33), we get that

$$\mathrm{ch}^\omega(J_w(\lambda)) = \sum_{\mu \in \Lambda^\omega} \mathrm{Tr}((\omega^{-1})^*|_{(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*)_\mu}) \, e(\mu),$$

where the linear operator $(\omega^{-1})^* \in GL(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*)$ is the transposed operator of $\omega^{-1} \in GL(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))$, which represents $\omega \in \langle \omega \rangle$ via the contragredient representation of $\langle \omega \rangle$ on the dual space $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*$. Here, by Corollary 3.1.3, we see that

$$\mathrm{Tr}(\omega|_{H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))_\mu}) \in \mathbb{Z}_{\geq 0} \quad \text{for all } \mu \in \Lambda^\omega.$$

In addition, for each $\mu \in \Lambda^\omega$, the μ-weight space of $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*$ is naturally identified with the dual space $\Big(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))_{-\mu}\Big)^*$. Hence the assertion immediately follows from Lemma 3.2.1 since $\omega^N = 1$. This completes the proof. $\square$

By combining Theorems 0.1 and 3.2.2, we obtain the following

Corollary 3.2.3. *Let* $\lambda \in \Lambda_+^\omega$ *and* $w \in W^\omega$. *Then we have in* $\mathbb{C}[\Lambda^\omega]$,

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\widehat{D}_w(e(\lambda))}.$$

Finally, by combining Corollary 3.1.3 and Theorem 3.2.2, we obtain a remarkable relation between the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of Joseph's module $J_w(\lambda)$ for $\mathfrak{g}$ and the ordinary character of Joseph's module for the orbit Lie algebra $\widehat{\mathfrak{g}}$. Namely, we have Corollary 0.2 in the introduction.

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