ON HEINZ'S INEQUALITY

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In memory of Professor Zygmunt Charzyński

Abstract. In 1958 E. Heinz obtained a lower bound for $|\partial_x F|^2 + |\partial_y F|^2$, where F is a one-to-one harmonic mapping of the unit disc onto itself keeping the origin fixed. We improve Heinz's inequality in the case where F is the Poisson integral of a sense-preserving homeomorphic self-mapping f of the unit circle. As an application we infer a version of Heinz's inequality for harmonic and quasiconformal self-mappings of the unit disc.

Introduction

Write $\mathrm{Hom}^+(\mathbb{T})$ for the class of all sense-preserving homeomorphic self-mappings of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Given a function $f : \mathbb{T} \to \mathbb{C}$ integrable on $\mathbb{T}$ we denote by $\mathrm{P}[f](z)$ the Poisson integral of f, i.e.

$$\mathrm{P}[f](z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \,\mathrm{Re}\, \frac{u+z}{u-z} |du| \ , \quad z \in \mathbb{D}\ , \tag{0.1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc. It is well known that the Jacobian $\mathrm{J}[\mathrm{P}[f]]$ is positive on $\mathbb{D}$ for every $f \in \mathrm{Hom}^+(\mathbb{T})$; see e.g. [1] or [4, p. 43]. Modifying considerations in [4, pp. 42-43] we obtain a stronger result given by Theorem 1.2 in Section 1. This implies [6, Remark 2.3] and thereby completes consideration in [6].

In 1958 E. Heinz proved that the inequality

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2} \tag{0.2}$$

holds for every $z = x + iy \in \mathbb{D}$, provided F is a one-to-one harmonic mapping of $\mathbb{D}$ onto itself and $F(0) = 0$; cf. [2]. Applying Theorem 1.2 and [6, Lemma 2.1], we are able to improve Heinz's inequality (0.2) in two cases. The first one, discussed in Section 2, deals with the case where $F = \mathrm{P}[f]$ for some $f \in \mathrm{Hom}^+(\mathbb{T})$; see Theorem 2.2. The second one, discussed in Section 3, deals with the case where F is a quasiconformal (qc. in abbreviation) mapping; see Theorem 3.2. The results were presented on Seminar: Generalized Cauchy-Riemann Structures and Surface Properties of Crystals, 23-30 July, 2001, Będlewo-Częstochowa, Poland.

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1. A lower estimate for the Jacobian

Given $f \in \mathrm{Hom}^+(\mathbb{T})$ and $z \in \mathbb{T}$ set

$$d_f := \operatorname*{ess\,inf}_{z\in\mathbb{T}} |f'(z)| \ , \tag{1.1}$$

where

$$f'(z) := \lim_{u\to z} \frac{f(u)-f(z)}{u-z} \tag{1.2}$$

provided the limit exists and $f'(z) := 0$ otherwise.

Lemma 1.1. *If $f \in \mathrm{Hom}^+(\mathbb{T})$, then $0 \le d_f \le 1$ and for every Borel subset $I \subset \mathbb{T}$,*

$$\sin \frac{|f(I)|_1}{2} \ge d_f \sin \frac{|I|_1}{2} \ , \tag{1.3}$$

where $|I|_1$ is the arc-length measure of I.

Proof. Let $m := d_f$. Obviously $m \ge 0$. If V is a Borel subset of $\mathbb{T}$, then

$$|f(V)|_1 \ge \int_V |f'(z)||dz| \ge m \int_V |dz| = m|V|_1 \ . \tag{1.4}$$

In particular $|\mathbb{T}|_1 = |f(\mathbb{T})|_1 \ge m|\mathbb{T}|_1$, and hence $m \le 1$. Applying (1.4) we obtain

$$|f(I)|_1 \ge m|I|_1 \quad \text{and} \quad |f(\mathbb{T}\setminus I)|_1 \ge m|\mathbb{T}\setminus I|_1 = m(2\pi - |I|_1) \ . \tag{1.5}$$

Since $f(\mathbb{T}\setminus I) = \mathbb{T}\setminus f(I)$ we conclude from (1.5) that

$$|f(I)|_1 = |\mathbb{T}\setminus(\mathbb{T}\setminus f(I)|_1 = 2\pi - |f(\mathbb{T}\setminus I)|_1 \le 2\pi - m(2\pi - |I|_1) \ , \tag{1.6}$$

and hence

$$m|I|_1 \le |f(I)|_1 \le 2\pi - m(2\pi - |I|_1) \ . \tag{1.7}$$

Since $0 \le |f(I)|_1/2 \le \pi$ we conclude from (1.7) that

$$\begin{aligned} \sin(|f(I)|_1/2) &\ge \min\{\sin(m|I|_1/2)\, , \sin(\pi - m(\pi - |I|_1/2))\} \\ &= \min\{\sin(m|I|_1/2)\, , \sin(m(\pi - |I|_1/2))\} \ . \end{aligned} \tag{1.8}$$

Since $\mathbb{R} \ni t \mapsto \sin t$ is a concave function on $[0;\pi]$, we have $\sin(mt) \ge m \sin t$ for $0 \le t \le \pi$. Thus

$$\sin(|f(I)|_1/2) \ge m \min\{\sin(|I|_1/2)\, , \sin(\pi - |I|_1/2))\} = m\sin(|I|_1/2) \tag{1.9}$$

follows from (1.8), which yields (1.3). $\square$

Theorem 1.2. *If $f \in \mathrm{Hom}^+(\mathbb{T})$, then*

$$\inf_{z\in\mathbb{D}} \mathrm{J}[\mathrm{P}[f]](z) \ge d_f^3 \ . \tag{1.10}$$

Proof. Given $h \in \mathrm{Hom}^+(\mathbb{T})$, $t \in \mathbb{R}$ and $s \in [0;\pi]$ define

$$\begin{aligned} h(t,s)_1 &:= |h(I(e^{it}, e^{i(t+s)}))|_1 \ , \\ h(t,s)_2 &:= |h(I(e^{i(t+s)}, e^{i(t+\pi)}))|_1 \ , \\ h(t,s)_3 &:= |h(I(e^{i(t+\pi)}, e^{i(t+s+\pi)}))|_1 \ , \\ h(t,s)_4 &:= |h(I(e^{i(t+s+\pi)}, e^{i(t+2\pi)}))|_1 \ , \end{aligned} \tag{1.11}$$

where $I(z,w)$ is a closed arc directed counterclockwise from $z\in\mathbb{T}$ to $w\in\mathbb{T}$. Following Douady and Earle [1] the Jacobian $\mathrm{J}[\mathrm{P}[h]](0)$ of h is equal to

$$\mathrm{J}[\mathrm{P}[h]](0)=\frac{1}{\pi^2}\int_0^{\pi}(\sin s\int_0^{2\pi}R_h(t,s)dt)ds\ , \tag{1.12}$$

where

$$R_h(t,s):=\sin\frac{h(t,s)_1+h(t,s)_2}{2}\sin\frac{h(t,s)_2+h(t,s)_3}{2}\sin\frac{h(t,s)_1+h(t,s)_3}{2}\ ;$$

see also [4, pp. 42-43]. For $a\in\mathbb{D}$ and $z\in\overline{\mathbb{D}}$ write

$$h_a(z):=\frac{z-a}{1-\overline{a}z}\ .$$

Fix $z\in\mathbb{D}$ and set

$$h(u):=f\circ h_{-z}(u)\ ,\quad u\in\mathbb{T}\ . \tag{1.13}$$

From (1.12) and (1.3) it follows that

$$\begin{aligned}\mathrm{J}[\mathrm{P}[h]](0)&=\frac{1}{\pi^2}\int_0^{\pi}(\sin s\int_0^{2\pi}R_{f\circ h_{-z}}(t,s)dt)ds\\&\geq\frac{d_f^3}{\pi^2}\int_0^{\pi}(\sin s\int_0^{2\pi}R_{h_{-z}}(t,s)dt)ds=d_f^3\,\mathrm{J}[\mathrm{P}[h_{-z}]](0)\ .\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{J}[\mathrm{P}[f]](z)&=\mathrm{J}[\mathrm{P}[h\circ h_z]](z)=\mathrm{J}[\mathrm{P}[h]\circ h_z](z)=\mathrm{J}[\mathrm{P}[h]](h_z(z))\,\mathrm{J}[h_z](z)\\&=\mathrm{J}[\mathrm{P}[h]](0)\,\mathrm{J}[h_z](z)\geq d_f^3\,\mathrm{J}[h_{-z}](0)\,\mathrm{J}[h_z](z)=d_f^3\,\mathrm{J}[h_{-z}](h_z(z))\,\mathrm{J}[h_z](z)\\&=d_f^3\,\mathrm{J}[h_{-z}\circ h_z](z)=d_f^3\,\mathrm{J}[\mathrm{id}](z)=d_f^3\ ,\end{aligned}$$

which proves (1.10). □

2. The case where F is given by the Poisson integral

Recall that the formal derivative operators ∂ and $\bar\partial$ are defined by the usual real partial derivatives ∂_x and ∂_y as below

$$\partial:=\frac{1}{2}(\partial_x-i\partial_y)\quad\text{and}\quad\bar\partial:=\frac{1}{2}(\partial_x+i\partial_y)\ . \tag{2.1}$$

Let $f\in\mathrm{Hom}^+(\mathbb{T})$. From [6, Lemma 2.1] it follows that for a.e. $z\in\mathbb{T}$ both the functions $\partial\,\mathrm{P}[f]$ and $\bar\partial\,\mathrm{P}[f]$ have radial limiting values at z and the following equalities hold

$$\begin{aligned}2z\lim_{r\to1^-}\partial\,\mathrm{P}[f](rz)&=\lim_{r\to1^-}\left[\frac{f(z)-\mathrm{P}[f](rz)}{1-r}+zf'(z)\right]\\2\overline{z}\lim_{r\to1^-}\bar\partial\,\mathrm{P}[f](rz)&=\lim_{r\to1^-}\left[\frac{f(z)-\mathrm{P}[f](rz)}{1-r}-zf'(z)\right]\ .\end{aligned} \tag{2.2}$$

Thus we may define

$$d_f^*:=\operatorname*{ess\,inf}_{z\in\mathbb{T}}\left|\lim_{r\to1^-}\partial\,\mathrm{P}[f](rz)\right|\ . \tag{2.3}$$

Following Heinz [2], we will prove the following lemma.

Lemma 2.1. *If $f \in \mathrm{Hom}^+(\mathbb{T})$ and if $F = \mathrm{P}[f]$, then*

$$\inf_{z\in\mathbb{D}} |\partial F(z)| \geq d_f^* \ . \tag{2.4}$$

Proof. From [5, (1.10)] it follows that

$$|\partial F(z)|^2 > \frac{1}{2\pi^2}\left(\frac{1-|a|}{1+|a|}\right)^2 > 0 \ , \quad z \in \mathbb{D} \ ,$$

where $a \in \mathbb{D}$ is a unique point satisfying $F(-a) = 0$. Hence the holomorphic function $1/\partial F$ on $\mathbb{D}$ belongs to the Hardy class H^∞, and so

$$\sup_{z\in\mathbb{D}} |\partial F(z)|^{-1} \leq \operatorname*{ess\,sup}_{z\in\mathbb{T}} |\lim_{r\to 1^-} \partial F(rz)|^{-1} \ .$$

Then (2.4) follows from (2.3), as claimed. □

Theorem 2.2. *If $f \in \mathrm{Hom}^+(\mathbb{T})$ and if $F := \mathrm{P}[f]$ satisfies $F(0) = 0$, then*

$$\inf_{z\in\mathbb{D}} |\partial F(z)|^2 \geq \frac{1}{\pi^2} + \frac{1}{4}d_f^2 + \frac{1}{4}\max\{d_f \ , \ 2d_f^3\} \tag{2.5}$$

and

$$\inf_{z\in\mathbb{D}} (|\partial_x F(z)|^2 + |\partial_y F(z)|^2) \geq \frac{2}{\pi^2} + \frac{1}{2}d_f^2 + \frac{1}{2}\max\{d_f \ , \ 2d_f^3\} \tag{2.6}$$

Proof. From (2.2) it follows that for a.e. $z \in \mathbb{T}$ the limits

$$\lim_{r\to 1^-} \frac{f(z) - F(rz)}{1-r} \quad \text{and} \quad \lim_{r\to 1^-} \mathrm{J}[F](rz)$$

exist and the following equalities hold:

$$2\lim_{r\to 1^-} (|\partial F(rz)|^2 + |\bar{\partial} F(rz)|^2) = |f'(z)|^2 + \lim_{r\to 1^-}\left|\frac{f(z) - F(rz)}{1-r}\right|^2 \tag{2.7}$$

as well as

$$\lim_{r\to 1^-} (|\partial F(rz)|^2 - |\bar{\partial} F(rz)|^2) = \lim_{r\to 1^-} \mathrm{J}[F](rz) \ . \tag{2.8}$$

Combining (2.7) with (2.8) we see that the equality

$$\lim_{r\to 1^-} |\partial F(rz)|^2 = \frac{1}{4}|f'(z)|^2 + \frac{1}{4}\lim_{r\to 1^-}\left|\frac{f(z) - F(rz)}{1-r}\right|^2 + \frac{1}{2}\lim_{r\to 1^-} \mathrm{J}[F](rz) \tag{2.9}$$

holds for a.e. $z \in \mathbb{T}$. Since F is harmonic on $\mathbb{D}$, $F(\mathbb{D}) = \mathbb{D}$ and $F(0) = 0$, we conclude from [2, Lemma] that

$$|F(z)| \leq \frac{4}{\pi}\arctan|z| \ , \quad z \in \mathbb{D} \ . \tag{2.10}$$

Actually, this is a version of Schwarz's lemma for harmonic self-mappings of $\mathbb{D}$. From (2.10) we see that for every $z \in \mathbb{T}$ and $r \in [0;1)$,

$$\left|\frac{f(z) - F(rz)}{1-r}\right| \geq \frac{|f(z)| - |F(rz)|}{1-r} \geq \frac{1 - \frac{4}{\pi}\arctan r}{1-r} \to \frac{2}{\pi} \quad \text{as } r \to 1^- \ . \tag{2.11}$$

By [6, Theorem 2.2] and by Theorem 1.2 we have

$$\lim_{r\to 1^-} \mathrm{J}[F](rz) \geq \frac{1}{2}\max\{d_f \ , \ 2d_f^3\} \quad \text{for a.e. } z \in \mathbb{T} \ .$$

Combining this with (2.9) and (2.11) we obtain

$$(d_f^*)^2 \geq \frac{1}{\pi^2} + \frac{1}{4}d_f^2 + \frac{1}{4}\max\{d_f \, , \, 2d_f^3\}$$

Thus Lemma 2.1 yields (2.5). Applying (2.1) we get

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 = 2(|\partial F(z)|^2 + |\bar{\partial} F(z)|^2) \; , \quad z \in \mathbb{D} \; . \tag{2.12}$$

Combining (2.5) with (2.12) we obtain (2.6), which completes the proof. □

3. The case where F is a quasiconformal mapping

It is well known that a quasiconformal self-mapping F of $\mathbb{D}$ has a homeomorphic extension F^* to the closure $\overline{\mathbb{D}}$; cf. [3]. We call the restriction $f := F^*_{|\mathbb{T}}$ the *boundary limiting valued function of* F. Suppose that F is additionally a harmonic mapping. Then $F = \mathrm{P}[f]$ on $\mathbb{D}$, as a unique solution to the Dirichlet problem with the boundary function f.

Lemma 3.1. *Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbb{D}$ satisfying $F(0) = 0$. If f is the boundary limiting valued function of F, then*

$$d_f \geq \frac{2}{\pi K} \; . \tag{3.1}$$

Proof. From (2.2) it follows that for a.e. $z \in \mathbb{T}$,

$$\begin{aligned} &\lim_{r\to 1^-} [z\partial F(rz) + \overline{z}\bar{\partial} F(rz)] = \lim_{r\to 1^-} \frac{f(z) - F(rz)}{1 - r} \\ &\lim_{r\to 1^-} [z\partial F(rz) - \overline{z}\bar{\partial} F(rz)] = zf'(z) \; . \end{aligned} \tag{3.2}$$

Since F is a K-quasiconformal mapping, we see from (3.2) that for a.e. $z \in \mathbb{T}$,

$$\begin{aligned} |f'(z)| &= \lim_{r\to 1^-} |z\partial F(rz) - \overline{z}\bar{\partial} F(rz)| \geq \lim_{r\to 1^-} (|\partial F(rz)| - |\bar{\partial} F(rz)|) \\ &\geq \frac{1}{K} \lim_{r\to 1^-} (|\partial F(rz)| + |\bar{\partial} F(rz)|) \geq \frac{1}{K} \lim_{r\to 1^-} (|z\partial F(rz) + \overline{z}\bar{\partial} F(rz)|) \\ &= \frac{1}{K} \lim_{r\to 1^-} \left| \frac{f(z) - F(rz)}{1 - r} \right| \; . \end{aligned}$$

Hence by (2.11) we deduce (3.1). □

Theorem 3.2. *Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbb{D}$ satisfying $F(0) = 0$. If f is the boundary limiting valued function of F, then the inequalities*

$$|\partial F(z)| \geq \frac{K+1}{K\pi} \tag{3.3}$$

and

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2}\left(1 + \frac{1}{K}\right)^2 \tag{3.4}$$

hold for every $z \in \mathbb{D}$.

Proof. Since F is a K-quasiconformal mapping, we have

$$(K+1)|\bar{\partial} F(w)| \leq (K-1)|\partial F(w)| \; , \quad w \in \mathbb{D} \; ,$$

and hence

$$2(K^2+1)|\partial F(w)|^2 \geq (K+1)^2(|\partial F(w)|^2 + |\bar{\partial} F(w)|^2)\ , \quad w \in \mathbb{D}\ . \tag{3.5}$$

Combining (2.7) with (3.1) and (2.11) we see that for a.e. $z \in \mathbb{T}$,

$$\lim_{r\to 1^-} (|\partial F(rz)|^2 + |\bar{\partial} F(rz)|^2) \geq \frac{2}{\pi^2 K^2} + \frac{2}{\pi^2} = \frac{2}{\pi^2}\left(1 + \frac{1}{K^2}\right)\ . \tag{3.6}$$

From this and (3.5) it follows that for a.e. $z \in \mathbb{T}$,

$$\lim_{r\to 1^-} |\partial F(rz)| \geq \frac{K+1}{\pi K}\ . \tag{3.7}$$

Applying now (2.4) we deduce (3.3). Then (3.4) follows directly from (3.3) and (2.12). □

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