LINK CORRESPONDING TO CLOSED 3-MANIFOLD

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0. Introduction

Let $\mathbb{L}$ be the set of the types $[L]$ of oriented links L in S^3, and $\mathbb{L}^{\mathrm{f}}$ the set of the types $[L,f]$ where L is an oriented link in S^3 and f is a finite regular coloring of L. Let $\mathbb{M}$ be the set of the types $[M]$ of closed connected oriented 3-manifolds M. By W. B. R. Lickorish [20] and A. H. Wallace [26], the Dehn surgery construction induces a surjective map which we call *the Dehn surgery map*

$$D^{\mathrm{f}} : \mathbb{L}^{\mathrm{f}} \longrightarrow \mathbb{M}$$

sending every colored link type $[L,f]$ to the type $[\chi(L,f)]$ of the Dehn surgery manifold $\chi(L,f)$. In this paper, we shall construct some faithful right inverses

$$A^{\mathrm{f}} : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{f}}$$

of the Dehn surgery map D^{f}. For this purpose, we shall observe that the set $\mathbb{L}$ is a well-ordered set by an order which we call the *braid order*. Let $\mathbb{L}^0$ be the subset of $\mathbb{L}^{\mathrm{f}}$ consisting of the type $[L,0]$ of a 0-colored link $(L,0)$. We shall construct a further special faithful right inverse

$$a^0_\infty : \mathbb{M} \longrightarrow \mathbb{L}^0$$

of the Dehn surgery map

$$D^0 = D^{\mathrm{f}}|_{\mathbb{L}^0} : \mathbb{L}^0 \longrightarrow \mathbb{M}$$

(which is also a surjective map) such that the oriented link L_M given by $a^0_\infty([M]) = [L_M, 0]$ is a hyperbolic link possibly with infinite volume and the exterior $E(L_M)$ determines the link L_M up to orientations. For this purpose, we study how to change the Dehn surgery description of a disconnected colored link by the topological imitation theory, which has been developed in [6,7,8,9,10,11,12,13,14,15,16,17]. We say that a coloring f of an oriented link L is *distinguished* if $[\chi(L,f')] \neq [\chi(L,f)]$ for every coloring f' of L with $f' \neq f$. We shall show that for every disconnected colored link (L, f_L) such that f_L is a finite regular coloring of L and $[\chi(L,f_L)] \neq [S^3]$, there is a normal imitation $q : (S^3, L^*) \longrightarrow (S^3, L)$ such that L^* is a totally hyperbolic link, $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$ and the colorings ∞ and $f_L q$ of L^* are distinguished (see Corollary 5.7). Writing $G_M = \pi_1(E(L_M))$, *we can assign to every closed connected oriented 3-manifold M a hyperbolic group G_M (that is a finitely generated torsion-free discrete subgroup of $PSL(2,\mathbb{C})$) so that any two closed connected oriented 3-manifolds M and M' are mutually homeomorphic if and only if the groups*

G_M and $G_{M'}$ are mutually isomorphic. The group G_M is abelian if and only if M is homeomorphic to S^3 or $S^1 \times S^2$, where we have $G_{S^3} \cong Z \oplus Z$ and $G_{S^1 \times S^2} \cong Z$ (see Theorem 5.10).

In §1, we introduce the braid order in the link type set $\mathbb{L}$ by which $\mathbb{L}$ is a well-ordered set. In §2, we review some terminologies on the Dehn surgery map. In §3, a general construction and some examples of faithful right inverses of the Dehn surgery map are made. We also explain here how to derive all the oriented 3-manifold invariants from oriented link invariants. In §4, we review some terminologies and some basic results of the topological imitation theory which are used in this paper. In §5, we discuss a distinguished coloring of a framed link in a 3-manifold to state the basic result (Theorem 5.3) on changing a Dehn surgery description by a normal imitation. Some consequences of Theorem 5.3 including the fundamental group version (Theorem 5.4), Corollary 5.7 and Theorem 5.10 cited above are proved here by assuming Theorem 5.3. The proof of Theorem 5.3 is given in §6.

1. The braid order in the types of oriented links

The *type* $[L]$ of an oriented link L in S^3 is the class of of an oriented link L' in S^3 such that there is an orientation-preserving homeomorphism $h : S^3 \longrightarrow S^3$ sending L' to L orientation-preservingly. As we stated in §0, we denote by $\mathbb{L}$ the set of types of oriented links in S^3. To define the braid order in $\mathbb{L}$, we first introduce an order (which we call the *permuting order*) in the set $\mathbb{Q}^r$ with r a fixed positive integer extending the usual order of the rational number set $\mathbb{Q}$. The *permuting minimal r-row* $\mathbf{c}_{\text{p-min}}$ of an r-row $\mathbf{c} = (c_1, c_2, \dots, c_r) \in \mathbb{Q}^r$ is the minimal r-row in the lexicographic order among all the r-rows obtained by permuting the coordinates c_i $(i = 1, 2, \dots, r)$ of $\mathbf{c}$. For two r-rows $\mathbf{c}, \mathbf{c}' \in \mathbb{Q}^r$, we define the *permuting order* $\mathbf{c} \prec \mathbf{c}'$ if either $\mathbf{c}_{\text{p-min}}$ is smaller than $\mathbf{c}'_{\text{p-min}}$ in the lexicographic order or $\mathbf{c}_{\text{p-min}} = \mathbf{c}'_{\text{p-min}}$ and $\mathbf{c}$ is smaller than $\mathbf{c}'$ in the lexicographic order. We see that the permuting order $\prec$ is a total order in $\mathbb{Q}^r$.

For every oriented link L in S^3, the Alexander theorem (see J. S. Birman [1]) says that there is a braid σ with $[\hat{\sigma}] = [L]$, where $\hat{\sigma}$ denotes the closure of the braid σ. The *braid index* $b(x)$ of an element $x \in \mathbb{L}$ is the minimum of the braid string number among all the braids σ with $[\hat{\sigma}] = x$. When we need to clarify the braid index b of the braid σ, we will use the notation $\sigma_{(b)}$ for σ. Let σ_i $(i = 1, 2, \dots, b-1)$ be the standard generators of the b-string braid group B_b. By convention, we regard the sign of the crossing point of the diagram σ_i as $+1$. If $b(x) = b$, then there is a braid $\sigma \in B_b$ with $[\hat{\sigma}] = x$, so that we can write

$$(*) \qquad \sigma = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \dots \sigma_{i_r}^{\epsilon_r}, \quad \epsilon_i = \pm 1 (i = 1, 2, \dots, r).$$

The *braid length* $\ell(x)$ of an element $x \in \mathbb{L}$ is the least number r among all the presentations (*) of all the braids $\sigma \in B_b$ with $b = b(x)$ and $[\hat{\sigma}] = x$. We note that $\ell(x) = 0$ if and only if $x = [\hat{1}_{(b)}]$ for the identity braid $1_{(b)} \in B_b$ with $b = b(x)$ if and only if x is the trivial link type with $b(x)$-components. The *enumeration index* $i(x)$ of an element $x \in \mathbb{L}$ with $\ell(x) = r > 0$ is the smallest r-row in the permuting order $\prec$ among the r-rows $(i_1, i_2, \ldots, i_r)$ for all the presentations (*) of all the braids $\sigma \in B_b$ with $b = b(x)$ and $[\hat{\sigma}] = x$. The *negative expotent* $e^-(x)$ of an element $x \in \mathbb{L}$ with $\ell(x) = r > 0$ is the smallest r-row in the permuting order $\prec$ among all the (-1)-multiple exponent r-rows $(-\epsilon_1, -\epsilon_2, \ldots, -\epsilon_r)$ of all the presentations (*) with $i(x) = (i_1, i_2, \ldots, i_r)$ of all $\sigma \in B_b$ with $[\hat{\sigma}] = x$ and $b = b(x)$. We see from the construction that the braid σ and the oriented link diagram $\hat{\sigma}$ are uniquely determined by the deta $b(x)$, $\ell(x)$, $i(x)$ and $e^-(x)$, and called the *braid* of x and the *oriented link diagram* of x, respectively.

For $x, y \in \mathbb{L}$, we define the order $x \prec y$ if we have either

(1) $b(x) < b(y)$,
(2) $b(x) = b(y)$ and $\ell(x) < \ell(y)$,
(3) $b(x) = b(y)$, $\ell(x) = \ell(y)$ and $i(x) \prec i(y)$, or
(4) $b(x) = b(y)$, $\ell(x) = \ell(y)$, $i(x) = i(y)$ and $e^-(x) \prec e^-(y)$.

This order $\prec$ is called the *braid order* of $\mathbb{L}$. Then we have the following lemma:

Lemma 1.1. The set $\mathbb{L}$ is a well-ordered set with respect to the braid order.

Proof. If $b(x) = b(y)$, $\ell(x) = \ell(y)$, $i(x) = i(y)$ and $e^-(x) = e^-(y)$, then we have $x = y$. Taking the contrapositive claim, we see that $\mathbb{L}$ is a totally ordered set with respect to the braid order $\prec$. Let $S(\mathbb{L})$ be a non-empty subset of $\mathbb{L}$. Let $S_b(\mathbb{L})$ be the subset of $S(\mathbb{L})$ consisting of elements with the smallest braid index among all $x \in S(\mathbb{L})$. Next, let $S_{b,\ell}(\mathbb{L})$ be the subset of $S_b(\mathbb{L})$ consisting of elements with the smallest braid length among all $x \in S_b(\mathbb{L})$. If the smallest braid length is 0, then $S_{b,\ell}(\mathbb{L})$ is a singleton set. Assume that the smallest braid length is positive. Then let $S_{b,\ell,i}(\mathbb{L})$ be the subset of $S_{b,\ell}(\mathbb{L})$ consisting of elements with the smallest enumeration index in the permuting order $\prec$ among all $x \in S_{b,\ell}(\mathbb{L})$. Next, let $S_{b,\ell,i,e^-}(\mathbb{L})$ be the subset of $S_{b,\ell,i}(\mathbb{L})$ consisting of elements with the smallest negative exponent in the permuting order $\prec$ among all $x \in S_{b,\ell,i}(\mathbb{L})$. Then the set $S_{b,\ell,i,e^-}(\mathbb{L})$ is the singleton set consisting of the initial element of the set $S(\mathbb{L})$ in the braid order, showing that $\mathbb{L}$ is a well-ordered set with respect to the braid order. □

2. The Dehn surgery map

We extensively consider as an oriented link a pair (M, L) where M is a compact connected oriented 3-manifold such that the boundary ∂M is empty or consists of tori and L is a locally flat closed oriented 1-submanifold. When the ambient manifold M is obvious, we denote the oriented link (M, L) by L. Let $K_i (i = 1, 2, \ldots, r)$ be the components of L. A meridian system $m(L)$ on a tubular neighborhood $N(L) = \cup_{i=1}^r N(K_i)$ of L in M is always defined as a system consisting of a meridian $m(K_i)$ of $N(K_i)$ for every $i = 1, 2, \ldots, r$. On the other hand, a longitude system $\ell(L)$ on $N(L)$ is not uniquely specified in general. A *framed* link is an oriented link (M,L) such that a longitude system $\ell(L)$ of L in M is specified on a tubular neighborhood $N(L)$ as a system consisting of a longitude $\ell(K_i)$ of $N(K_i)$ for every $i=1,2,\dots,r$. By a *meridian-longitude system* of a framed link L, we mean a pair of a meridian system $m(L)$ and a longitude system $\ell(L)$ on $N(L)$ such that $m(K_i)$ meets $\ell(K_i)$ transversely in a single point for every i. We can specify the orientations of $m(L)$ and $\ell(L)$ from those of L and M uniquely. When $M=S^3$, we have a *canonical* meridian-longitude system $(m(L),\ell(L))$ of L by taking a canonical longitude $\ell(K_i)$ on $N(K_i)$ characterized by that $\ell(K_i)$ is null-homologous in the exterior $E(K_i)=\mathrm{cl}(S^3-N(K_i)$. Unless otherwise stated, we will consider an oriented link L in S^3 as a framed link by taking a canonical meridian-longitude system of L.

Definition 2.1. A *coloring* f of a framed link L is a map

$$f:\{K_i|\, i=1,2,\dots,r\} \longrightarrow \mathbb{Q}^+,$$

where $\mathbb{Q}^+=\mathbb{Q}\cup\{\infty,\emptyset\}$ for symbols ∞, $\emptyset$ with the identities $-\infty=\infty$, $-\emptyset=\emptyset$, $\infty+c=c+\infty=\infty$ and $\emptyset+c=c+\emptyset=\emptyset$ $(c\in\mathbb{Q})$.

Let $f(L)$ be the subset of $\mathbb{Q}^+$ consisting of the element $f(K_i)\in\mathbb{Q}^+$ for all i. A coloring f of L is the *c-coloring* for an element $c\in\mathbb{Q}^+$ if f is the constant map to c, i.e., $f(K_i)=c$ for all i. A colored link (L',f') is *equivalent* to a colored link (L,f) if there is an orientation-preserving homeomorphism $h:M\longrightarrow M$ sending L to L' orientation-preservingly such that $f(K_i)=f'h(K_i)$ for all i. The type $[L,f]$ of a colored link (L,f) is the class of colored links equivalent to (L,f). A coloring f of L is *finite* if $f(L)\subset\mathbb{Q}\cup\{\emptyset\}$, and *regular* if $f(L)\subset\mathbb{Q}\cup\{\infty\}$. By $\mathbb{L}^{\mathrm{f}}$, we denote the set of the colored link types $[L,f]$ such that L is an oriented link in S^3 and f is a finite regular coloring of L. The *size* of a rational number $c=\frac{a}{b}$ with a,b coprime integers is the integer $\rho(c)=|a|+|b|$. The sizes of the symbols $\emptyset$ and ∞ are $\rho(\emptyset)=0$ and $\rho(\infty)=1$ by convention. The *size* $\rho(f)$ of a coloring f of L is the set of the sizes $\rho(f(K_i))$ for all i. For an integer J, if we have $\rho(f(K_i))\geq J$ for all i or $\rho(f(K_i))>J$ for all i, respectively, then we denote it by $\rho(f)\geq J$ or $\rho(f)>J$, respectively. Similarly, if we have $\rho(f(K_i))\leq J$ for all i or $\rho(f(K_i))<J$ for all i, respectively, then we denote it by $\rho(f)\leq J$ or $\rho(f)<J$, respectively.

By re-indexing the components K_i $(i=1,2,\dots,r)$, let K_i $(i=1,2,\dots,u)$ be the components of L with $f(K_i)\neq\emptyset$. Let $f(K_i)=\frac{a_i}{b_i}$ for coprime integers a_i,b_i where we take $a_i=\pm1$ and $b_i=0$ when $f(K_i)=\infty$. Then we have a (unique up to isotopies) simple loop s_i on $\partial N(K_i)$ with $[s_i]=a_i[m_i]+b_i[\ell_i]$ in $H_1(\partial N(K_i);Z)$ for the meridian-longitude pair (m_i,ℓ_i) of K_i on $N(K_i)$. We note that if the different choice $f(K_i)=\frac{-a_i}{-b_i}$ is made, then only the orientation of s_i is changed.

Definition 2.2. The *Dehn surgery manifold* of a colored link (L,f) is the oriented

3-manifold

$$\chi(L,f)=E(L)\cup_{s_1=1\times\partial D_1^2}S^1\times D_1^2\cdots\cup_{s_r=1\times\partial D_r^2}S^1\times D_u^2$$

with the orientation induced from $E(L)\subset M$, where $\cup_{s_i=1\times\partial D_i^2}$ denotes a pasting of $S^1\times\partial D_i^2$ to $\partial N(K_i)$ so that s_i is identified with $1\times\partial D_i^2$.

By definition, we have $\chi(L,f)=E(L)$ if $f=\emptyset$ and $\chi(L,f)=M$ if $f=\infty$. The *type* $[M]$ of an oriented 3-manifold M is the class of oriented 3-manifolds which are orientation-preservingly homeomorphic to M. The set of types of closed connected oriented 3-manifolds is denoted by $\mathbb{M}$. In the construction of $\chi(L,f)$, the type $[\chi(L,f)]$ is independent of choices of orientations of the simple loops s_i and hence determined uniquely from the colored link type $[L,f]$. We denote by $\mathbb{L}^{\mathrm{f}}$ the set of types $[L,f]$ where L is an oriented link in S^3 and f is a finite regular coloring of L. The *Dehn surgery map*

$$D^{\mathrm{f}}:\mathbb{L}^{\mathrm{f}}\longrightarrow\mathbb{M}$$

is a map sending $[L,f]$ to $[\chi(L,f)]$. This map D^{f} is well-known to be surjective by W. B. R. Lickorish [20] and A. H. Wallace [26]. Let $\mathbb{L}^0$ be the subset of $\mathbb{L}^{\mathrm{f}}$ consisting of 0-colored link types $[L,0]$. The following lemma is a folklore result obtained by the Kirby calculus (see R. Kirby [19]):

Lemma 2.3. The restriction

$$D^0=D^{\mathrm{f}}|_{\mathbb{L}^0}:\mathbb{L}^0\longrightarrow\mathbb{M}$$

of the Dehn surgery map D^{f} to $\mathbb{L}^0$ is a surjection.

Proof. For every $[M]\in\mathbb{M}$, we have a colored link (L,f) with the components K_i $(i=1,2,\ldots,r)$ such that $[\chi(L,f)]=[M]$ and $f(K_i)=m_i$ is an even integer for all i (see S. J. Kaplan [5]). Let $L_1=L\cup L_0$ be the split union of the oriented link L and a negative Hopf link L_0. Let f_1 be the coloring of L_1 obtained from f and the 0-coloring of L_0. If $\mathrm{sign}(m_i)=+1$, then we take a fusion K_i' of K_i and $\frac{|m_i|}{2}$ paralell copies of L_0. If $\mathrm{sign}(m_i)=-1$, then we take a fusion K_i' of K_i and $\frac{|m_i|}{2}$ paralell copies of L_0 with the orientations of all the paralell copies of one component of L_0 reversed. Replacing K_i with K_i' for all i with $m_i\neq 0$, we obtain an oriented link L_1' from L_1 such that $[\chi(L_1',0)]=[\chi(L_1,f_1)]=[M]$. □

3. Faithful right inverses of the Dehn surgery map

In this section, we consider how to construct faithful right inverses of the Dehn surgery map $D^{\mathrm{f}}:\mathbb{L}^{\mathrm{f}}\longrightarrow\mathbb{M}$. Let

$$\pi:\mathbb{L}^{\mathrm{f}}\longrightarrow\mathbb{L}$$

be the forgetful surjective map sending every colored link type $x^{\mathrm{f}} = [L, f] \in \mathbb{L}^{\mathrm{f}}$ to the link type $x = [L] \in \mathbb{L}$. We define $b(x^{\mathrm{f}}) = b(x)$, $\ell(x^{\mathrm{f}}) = \ell(x)$, $i(x^{\mathrm{f}}) = i(x)$ and $e^{-}(x^{\mathrm{f}}) = e^{-}(x)$. Let σ be the braid of x. We can consider the oriented link diagram $L = \hat{\sigma}$ of x as an ordered link by the order among the components of L appearing first as a braid string of the braid σ. Let K_i $(i = 1, 2, \ldots, r)$ be the components of L with K_i the ith component ordered in this way. Let $f(L) = (f(K_1), f(K_2), \ldots, f(K_r)) \in \mathbb{Q}^r$ for a finite regular coloring f of L. The *size*, *absolute value* and *negative sign* of an r-row $\mathbf{c} = (c_1, c_2, \ldots, c_r) \in \mathbb{Q}^r$ are respectively the r-rows

$$\begin{aligned} \rho(\mathbf{c}) &= (\rho(c_1), \rho(c_2), \ldots, \rho(c_r)), \\ |\mathbf{c}| &= (|c_1|, |c_2|, \ldots, |c_r|), \\ \mathrm{sign}^{-}(\mathbf{c}) &= (-\mathrm{sign}(c_1), -\mathrm{sign}(c_2), \ldots, -\mathrm{sign}(c_r)), \end{aligned}$$

where the sign of a rational number c is defined by

$$\mathrm{sign}(c) = \begin{cases} \frac{c}{|c|} & (c \neq 0) \\ 0 & (c = 0). \end{cases}$$

Let $\mathcal{F}$ be the set of finite regular colorings f of L such that $[L, f] = x^{\mathrm{f}}$. The *size* $\rho(x^{\mathrm{f}})$ of x^{f} is the smallest r-row in the permuting order $\prec$ among the r-rows $\rho(f(L))$ for all $f \in \mathcal{F}$. Since the permuting minimal r-rows $\rho(f(L))_{\text{p-min}}$ are the same for all $f \in \mathcal{F}$, the size $\rho(x^{\mathrm{f}})$ is equal to the smallest r-row in the lexicographic order among the r-rows $\rho(f(L))$ for all $f \in \mathcal{F}$. Let $\mathcal{F}'$ be the subset of $\mathcal{F}$ consisting a finite regular coloring f' of L with $\rho(f'(L)) = \rho(x^{\mathrm{f}})$. The *absolute value* $|x^{\mathrm{f}}|$ of x^{f} is the smallest r-row in the permuting order $\prec$ among the r-rows $|f'(L)|$ for all $f' \in \mathcal{F}'$ which is equal to the smallest r-row in the lexicographic order. Let $\mathcal{F}''$ be the subset of $\mathcal{F}'$ consisting a finite regular coloring f'' of L with $|f''(L)| = |x^{\mathrm{f}}|$. The *negative sign* $\mathrm{sign}^{-}(x^{\mathrm{f}})$ of x^{f} is the smallest r-row in the permuting order $\prec$ among the r-rows $\mathrm{sign}^{-}(f''(L))$ for all $f'' \in \mathcal{F}''$ which is equal to the smallest r-row in the lexicographic order. Let $\mathcal{F}'''$ be the subset of $\mathcal{F}''$ consisting a finite regular coloring f''' of L with $\mathrm{sign}^{-}(f'''(L)) = \mathrm{sign}^{-}(x^{\mathrm{f}})$. Then we see that $\mathcal{F}'''$ is a singleton set. Thus, two colored link types $x^{\mathrm{f}}, y^{\mathrm{f}} \in \mathbb{L}^{\mathrm{f}}$ are equal if and only if $x = \pi(x^{\mathrm{f}}) = \pi(y^{\mathrm{f}}) = y$, $\rho(x^{\mathrm{f}}) = \rho(y^{\mathrm{f}})$, $|x^{\mathrm{f}}| = |y^{\mathrm{f}}|$ and $\mathrm{sign}^{-}(x^{\mathrm{f}}) = \mathrm{sign}^{-}(y^{\mathrm{f}})$.

For $x^{\mathrm{f}}, y^{\mathrm{f}} \in \mathbb{L}^{\mathrm{f}}$, we define the order $x^{\mathrm{f}} \prec y^{\mathrm{f}}$ which we also call the *braid order* in $\mathbb{L}^{\mathrm{f}}$ if we have either

(1) $x = \pi(x^{\mathrm{f}}) \prec \pi(y^{\mathrm{f}}) = y$,
(2) $x = y$, $\rho(x^{\mathrm{f}}) \prec \rho(y^{\mathrm{f}})$,
(3) $x = y$, $\rho(x^{\mathrm{f}}) = \rho(y^{\mathrm{f}})$ and $|x^{\mathrm{f}}| \prec |y^{\mathrm{f}}|$, or
(4) $x = y$, $\rho(x^{\mathrm{f}}) = \rho(y^{\mathrm{f}})$, $|x^{\mathrm{f}}| = |y^{\mathrm{f}}|$ and $\mathrm{sign}^{-}(x^{\mathrm{f}}) \prec \mathrm{sign}^{-}(y^{\mathrm{f}})$.

Then we have the following lemma:

Lemma 3.1. The set $\mathbb{L}^{\mathrm{f}}$ is a well-ordered set with respect to the braid order.

Proof. If $x^{\mathrm{f}} \neq y^{\mathrm{f}}$, then we see that $x^{\mathrm{f}} \prec y^{\mathrm{f}}$ or $y^{\mathrm{f}} \prec x^{\mathrm{f}}$, so that $\mathbb{L}^{\mathrm{f}}$ is a totally ordered set. Let $S(\mathbb{L}^{\mathrm{f}})$ be a non-empty subset of $\mathbb{L}^{\mathrm{f}}$. Let x be the initial element of the subset $\pi(S(\mathbb{L}^{\mathrm{f}})) \subset \mathbb{L}$, and $S(\mathbb{L}^{\mathrm{f}})_{\pi} = \pi^{-1}(x) \cap S(\mathbb{L}^{\mathrm{f}})$ which is a non-empty subset of $S(\mathbb{L}^{\mathrm{f}})$. Let $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho}$ be the subset of $S(\mathbb{L}^{\mathrm{f}})_{\pi}$ consisting of an element x^{f} with the smallest size $\rho(x^{\mathrm{f}})$ in the permuting order $\prec$ among all $x^{\mathrm{f}} \in S(\mathbb{L}^{\mathrm{f}})_{\pi}$. Then we see that the set $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho}$ is a finite set. Next, let $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho,||}$ be the subset of $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho}$ consisting of an element x^{f} with the smallest absolute value $|x^{\mathrm{f}}|$ in the permuting order $\prec$ among all $x^{\mathrm{f}} \in S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho}$. Finally, let $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho,||,\mathrm{sign}^-}$ be the subset of $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho,||}$ consisting of an element x^{f} with the smallest negative sign $\mathrm{sign}^-(x^{\mathrm{f}})$ in the permuting order $\prec$ among all $x^{\mathrm{f}} \in S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho,||}$. Then $S(\mathbb{L}^{\mathrm{f}})_{\pi,\rho,||,\mathrm{sign}^-}$ is the singleton set, showing that $\mathbb{L}^{\mathrm{f}}$ is a well-ordered set with respect to the braid order. □

Let $\mathcal{P}(\mathbb{L}^{\mathrm{f}})$ be the set of non-empty subsets of $\mathbb{L}^{\mathrm{f}}$. By Lemma 3.1, we have a map

$$\psi : \mathcal{P}(\mathbb{L}^{\mathrm{f}}) \longrightarrow \mathbb{L}^{\mathrm{f}}$$

sending every $S(\mathbb{L}^{\mathrm{f}}) \in \mathcal{P}(\mathbb{L}^{\mathrm{f}})$ to the initial element of $S(\mathbb{L}^{\mathrm{f}})$ in the braid order, which we call the *braid choice function*. Let $\overline{x^{\mathrm{f}}} = [-\bar{L}, -f] \in \mathbb{L}^{\mathrm{f}}$ for every element $x^{\mathrm{f}} = [L, f] \in \mathbb{L}^{\mathrm{f}}$, where $-\bar{L}$ denotes the mirror image $\bar{L}$ of L with opposite orientation. Let $\bar{x} = [-M] \in \mathbb{M}$ for every element $x = [M] \in \mathbb{M}$. A right inverse

$$A : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{f}}$$

of the Dehn surgery map D^{f} is said to be *faithful* if we have $A(\bar{x}) = \overline{A(x)}$ for every $x \in \mathbb{M}$ with $x \neq \bar{x}$. Let

$$\overline{S(\mathbb{L}^{\mathrm{f}})} = \{\overline{x^{\mathrm{f}}} \,|\, x^{\mathrm{f}} \in S(\mathbb{L}^{\mathrm{f}})\} \in \mathcal{P}(\mathbb{L}^{\mathrm{f}})$$

for every $S(\mathbb{L}^{\mathrm{f}})\} \in \mathcal{P}(\mathbb{L}^{\mathrm{f}})$. Then we have the following criterion to constructing a faithful right inverse of the Dehn surgery map D^{f}:

Criterion 3.2. Assume that there is a map

$$\mathcal{A}^{\mathrm{f}} : \mathbb{M} \longrightarrow \mathcal{P}(\mathbb{L}^{\mathrm{f}})$$

such that $D^{\mathrm{f}}\mathcal{A}^{\mathrm{f}}(x) = \{x\}$ for every $x \in \mathbb{M}$ and $\mathcal{A}^{\mathrm{f}}(\bar{x}) = \overline{\mathcal{A}^{\mathrm{f}}(x)}$ for every $x \in \mathbb{M}$ with $x \neq \bar{x}$. Then the braid choice function ψ and the map $\mathcal{A}^{\mathrm{f}}$ define a unique faithful right inverse

$$A^{\mathrm{f}} : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{f}}$$

of the Dehn surgery map D^{f} such that

(1) $A^{\mathrm{f}}(x) \in \mathcal{A}^{\mathrm{f}}(x)$ for every $x \in \mathbb{M}$,
(2) $A^{\mathrm{f}}(x) = \psi\mathcal{A}^{\mathrm{f}}(x)$ if $x = \bar{x}$, and $\psi(\mathcal{A}^{\mathrm{f}}(x) \cup \mathcal{A}^{\mathrm{f}}(\bar{x}))$ is equal to either $A^{\mathrm{f}}(x)$ or $A^{\mathrm{f}}(\bar{x})$ if $x \neq \bar{x}$.

Proof. If $x \neq \bar{x}$, then $\psi\mathcal{A}^{\mathrm{f}}(x) \neq \psi\mathcal{A}^{\mathrm{f}}(\bar{x})$ and $\psi(\mathcal{A}^{\mathrm{f}}(x) \cup \mathcal{A}^{\mathrm{f}}(\bar{x}))$ is equal to either $\psi\mathcal{A}^{\mathrm{f}}(x)$ or $\psi\mathcal{A}^{\mathrm{f}}(\bar{x})$ which we call $A^{\mathrm{f}}(x)$ or $A^{\mathrm{f}}(\bar{x})$, respectively. Since $\mathcal{A}^{\mathrm{f}}(\bar{x}) = \overline{\mathcal{A}^{\mathrm{f}}(x)}$, we have the desired unique faithful right inverse $A^{\mathrm{f}} : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{f}}$. □.

The following example may be useful in making a table of closed oriented 3-manifolds:

Example 3.3. For every element $x \in \mathbb{M}$ we take the subset $\mathbb{L}^{\mathrm{f}}(x) \subset \mathbb{L}^{\mathrm{f}}$ consisting of a colored link type $x^{\mathrm{f}} = [L, f]$ such that $D^{\mathrm{f}}(x^{\mathrm{f}}) = [\chi(L, f)] = x$. By W. B. R. Lickorish [20] and A. H. Wallace [26], $\mathbb{L}^{\mathrm{f}}(x)$ is not empty. Since $D^{\mathrm{f}}\mathbb{L}^{\mathrm{f}}(x) = \{x\}$ and $\mathbb{L}^{\mathrm{f}}(\bar{x}) = \overline{\mathbb{L}^{\mathrm{f}}(x)}$ for every $x \in \mathbb{M}$, we obtain from Criterion 3.2 a unique faithful right inverse

$$a^{\mathrm{f}} : \mathbb{M} \longrightarrow \mathbb{L}^{\mathrm{f}}$$

of the Dehn surgery map D^{f} such that $a^{\mathrm{f}}(x) \in \mathbb{L}^{\mathrm{f}}(x)$ for every $x \in \mathbb{M}$ and $a^{\mathrm{f}}(x) = \psi\mathbb{L}^{\mathrm{f}}(x)$ if $x = \bar{x}$ and $\psi(\mathbb{L}^{\mathrm{f}}(x) \cup \mathbb{L}^{\mathrm{f}}(\bar{x}))$ is equal to either $a^{\mathrm{f}}(x)$ or $a^{\mathrm{f}}(\bar{x})$ if $x \neq \bar{x}$. For example, we have the following calculations whose proofs are good exercise of the Kirby calculus (cf. D. Rolfsen [22]):

(3.3.1) $a^{\mathrm{f}}([S^3]) = [\widehat{1_{(1)}}, 1]$ and $a^{\mathrm{f}}([S^1 \times S^2]) = [\widehat{1_{(1)}}, 0]$

(3.3.2) We have $a^{\mathrm{f}}([L(p,q)] = [\widehat{1_{(1)}}, \frac{p}{q}]$ for a suitably oriented lens space $L(p,q)$ $(0 < q \leq \frac{p}{2})$.

(3.3.3) For a suitably oriented quotanion space Q, we have $a^{\mathrm{f}}[Q]) = [\widehat{\sigma^4_{1(2)}}, 0]$.

(3.3.4) For the 3-torus $T^3 = S^1 \times S^1 \times S^1$, we have $a^{\mathrm{f}}([T^3]) = [\widehat{\sigma}, 0]$ with $\sigma = (\sigma_1\sigma_2^{-1})^3_{(3)}$.

(3.3.5) For a suitably oriented Poincaré homology 3-sphere M, we have $a^{\mathrm{f}}([M]) = [\widehat{\sigma^3_{1(2)}}, 1]$.

Since $\mathbb{L}^{\mathrm{f}}$ is a well-ordered set by the braid order, $\mathbb{M}$ is a well-ordered set with respect to the order given by the following definition:

$$x \prec y \Longleftrightarrow a^{\mathrm{f}}(x) \prec a^{\mathrm{f}}(y)$$

for $x, y \in \mathbb{M}$. The enumeration of the 3-manifolds in (3.3.1)-(3.3.5) in this order is as follows:

$$[S^1 \times S^2] \prec [S^3] \prec [L(p,q)] \prec [M] \prec [Q] \prec [T^3].$$

The following example concerns a correspondence from $\mathbb{M}$ to $\mathbb{L}$:

Example 3.4. Let $\mathbb{L}^0$ be the subset of $\mathbb{L}$ consisting of 0-colored link types. For every $x \in \mathbb{M}$ we take the subset $\mathbb{L}^0(x) \subset \mathbb{L}^0$ consisting of a 0-colored link type

$x^0 = [L, 0]$ such that $D^0(x^0) = D^{\rm f}(x^0) = [\chi(L, 0)] = x$. By Lemma 2.3, $\mathbb{L}^0(x)$ is not empty. Since $D^{\rm f}\mathbb{L}^0(x) = \{x\}$ and $\mathbb{L}^0(\bar{x}) = \overline{\mathbb{L}^0(x)}$ for every $x \in \mathbb{M}$, we obtain from Criterion 3.2 a unique faithful right inverse

$$a^0 : \mathbb{M} \longrightarrow \mathbb{L}^0$$

of the Dehn surgery map D^0 such that $a^0(x) \in \mathbb{L}^0(x)$ for every $x \in \mathbb{M}$ and $a^0(x) = \psi\mathbb{L}^0(x)$ if $x = \bar{x}$ and $\psi(\mathbb{L}^0(x) \cup \mathbb{L}^0(\bar{x}))$ is equal to either $a^0(x)$ or $a^0(\bar{x})$ if $x \neq \bar{x}$. Incidentally, it would be interesting to know *whether or not there is an element* $x^0 \in \mathbb{L}^0(x)$ *with* $\overline{x^0} = x^0$ *for every element* $x \in \mathbb{M}$ *with* $\bar{x} = x$. Using the identification map $\mathbb{L}^0 \longrightarrow \mathbb{L}$ sending $[L, 0]$ to $[L]$, we can identify the maps D^0 and a^0 with the maps

$$D : \mathbb{L} \longrightarrow \mathbb{M} \quad \text{and} \quad a : \mathbb{M} \longrightarrow \mathbb{L}$$

respectively such that a is a right inverse of D with $a(\bar{x}) = \overline{a(x)}$ for every $x \in \mathbb{M}$ with $x \neq \bar{x}$. Some calculations are made as follows:

(3.4.1) $a[S^3] = [\widehat{\sigma^2_{1(2)}}]$ which is the positive Hopf link type.
(3.4.2) $a[S^1 \times S^2] = [\widehat{1_{(1)}}]$ which is the trivial knot type.
(3.4.3) For the projective 3-space $P^3 = L(2, 1)$, we have $a([P^3]) = [\widehat{\sigma}]$ with $\sigma = (\sigma_1\sigma_2)^3_{(3)}$.
(3.4.4) For a suitably oriented quotanion space Q, we have $a([Q]) = [\widehat{\sigma^4_{1(2)}}]$.
(3.4.5) For the 3-torus $T^3 = S^1 \times S^1 \times S^1$, we have $a([T^3]) = [\widehat{\sigma}]$ with $\sigma = (\sigma_1\sigma_2^{-1})^3_{(3)}$.
(3.4.6) For any oriented Poincaré homology 3-sphere M, the oriented link L_M has at least 10 components.

The proofs of (3.4.1)-(3.4.5) are exercise of the Kirby calculus (cf. D. Rolfsen [22]). To see (3.4.6), assume that L_M has r components. Using that M is a homology 3-sphere and $[M] = [\chi(L_M, 0)]$, we see that M bounds a simply connected 4-manifold W with an $r \times r$ non-singular intersection matrix whose diagonal entries are all 0. Since the Rochlin invariant of M is non-trivial, it follows that the signature of W is an odd multiple of 8 and r is even. Hence $m \geq 8$. If $m = 8$, then the intersection matrix is a positive or negative definite matrix which is not in our case. Thus, we have $m \geq 10$, showing (3.4.6).

The set $\mathbb{M}$ is also a well-ordered set with respect to the order given by the following definition:

$$x \prec y \Longleftrightarrow a(x) \prec a(y)$$

for $x, y \in \mathbb{M}$. The enumeration of the manifolds in (3.4.1)-(3.4.6) in this order is as follows:

$$[S^1 \times S^2] \prec [S^3] \prec [Q] \prec [P^3] \prec [T^3] \prec [M].$$

By *an oriented 3-manifold invariant* and *an oriented link invariant*, we mean maps

$$\mathbb{M} \longrightarrow \Lambda \quad \text{and} \quad \mathbb{L} \longrightarrow \Lambda$$

respectively, where Λ denotes an algebraic system. Let $\mathrm{Inv}(\mathbb{M},\Lambda)$ and $\mathrm{Inv}(\mathbb{L},\Lambda)$ be the sets of oriented 3-manifold invariants and oriented link invariants, respectively. By the right inverese a of D in Example 3.4, we have the following sequence

$$\mathrm{Inv}(\mathbb{M},\Lambda) \xrightarrow{D^{\#}} \mathrm{Inv}(\mathbb{L},\Lambda) \xrightarrow{a^{\#}} \mathrm{Inv}(\mathbb{M},\Lambda)$$

of the dual maps $a^{\#}$ and $D^{\#}$ of a and D. Since the composite $a^{\#}D^{\#}$ is the identity map on $\mathrm{Inv}(\mathbb{M},\Lambda)$, we see that the subset $D^{\#}(\mathrm{Inv}(\mathbb{M},\Lambda))$ of the set $\mathrm{Inv}(\mathbb{L},\Lambda)$ of oriented link invariants is mapped bijectively by $a^{\#}$ onto the set $\mathrm{Inv}(\mathbb{M},\Lambda)$ of oriented 3-manifold invariants, where this bijective map is independent of $a^{\#}$. For example, the Witten invariant $\tau_r \in \mathrm{Inv}(\mathbb{M},\mathbb{C})$ (see E. Witten [27]) induces an oriented link invariant $D^{\#}(\tau_r) \in \mathrm{Inv}(\mathbb{L},\mathbb{C})$ with $a^{\#}(D^{\#}(\tau_r)) = \tau_r$.

On the other hand, the surjection

$$a^{\#} : \mathrm{Inv}(\mathbb{L},\Lambda) \longrightarrow \mathrm{Inv}(\mathbb{M},\Lambda)$$

depends on the right inverse a of D, which implies that constructing a nice right inverse of D would lead to a nice oriented 3-manifold invariant. An example of a right inverse of D similar to but different from the right inverse a which leads to a hyperbolic group classification of closed connected oriented 3-manifolds stated in the introduction is given from this viewpoint in §5. Here is one example on creating an oriented 3-manifold invariant from an oriented link invariant.

Example 3.5. Let $\lambda \in \mathrm{Inv}(\mathbb{L},\mathbb{Z})$ be the signature invariant which is the signature of the symmetric matrix $V + V'$ for a Seifert matrix V associated with a connected Seifert surface of an oriented link (see [18]). The right inverse a of $D : \mathbb{L} \longrightarrow \mathbb{M}$ in Example 3.4 induces the oriented 3-manifold invariant

$$\lambda_a = a^{\#}(\lambda) \in \mathrm{Inv}(\mathbb{M},\mathbb{Z}).$$

For the 3-manifolds in (3.4.1)-(3.4.5), this invariant is calculated as follows:

(3.5.1) $\lambda_a(S^3) = -1$.
(3.5.2) $\lambda_a(S^1 \times S^2) = 0$.
(3.5.3) $\lambda_a(P^3) = -4$.
(3.5.4) $\lambda_a(\pm Q) = \mp 3$, where we note that $[Q] \neq [-Q]$.
(3.5.5) $\lambda_a(T^3) = 0$.

4. Review on AID and normal imitations

In this section, we briefly explain some notions of imitations of a link (M, L) where M is a compact connected oriented 3-manifold witout boundary or with only torus boundary components. See $[6-17]$ for more detailed accounts. In this section, we grant the link L to be empty unless otherwise stated. Let $I = [-1, 1]$. The concept of topological imitation arose from an interpretation of *reflection*. Namely, for an oriented link (M, L), an involution α on $(M, L) \times I = (M \times I, L \times I)$ is called a *reflection* in $(M, L) \times I$ if

(1) $\alpha((M, L) \times 1) = (M, L) \times (-1)$, and

(2) the fixed point set $\mathrm{Fix}(\alpha, (M, L) \times I)$ of α in $(M, L) \times I$ is an oriented link.

The reflection α is *standard* if $\alpha(x, t) = (x, -t)$ for all $(x, t) \in M \times I$, and *normal* if $\alpha(x, t) = (x, -t)$ for all $(x, t) \in \partial(M \times I) \cup N(L) \times I$ for a tubular neighborhood $N(L)$ of L in M. The reflection α is *isotopically standard* if $h^{-1}\alpha h$ is standard for an auto-homeomorphism h of $M \times I$ which is isotopic to the identity by an isotopy keeping $\partial(M \times I) \cup N(L) \times I$ fixed for a tubular neighborhood $N(L)$ of L in M. Further, the reflection α is *isotopically almost standard* if $L \neq \emptyset$ and α defines an isotopically standard reflection in $(M, L - K) \times I$ for every component K of L. A *reflector* of a reflection α in $(M, L) \times I$ is an embedding

$$\phi_\alpha : (M^*, L^*) \longrightarrow (M, L) \times I$$

with $\phi_\alpha(M^*, L^*) = \mathrm{Fix}(\alpha, (M, L) \times I)$.

Definition 4.1. An *imitation* of (M, L) is the composite

$$q : (M^*, L^*) \xrightarrow{\phi_\alpha} (M, L) \times I \xrightarrow{\mathrm{proj}} (M, L)$$

where $\phi_\alpha : (M^*, L^*) \longrightarrow (M, L) \times I$ is reflector of a reflection α in $(M, L) \times I$.

We also call (M^*, L^*) an *imitation* of (M, L) (with *imitation map* q). We note that the pair (M^*, L^*) is also an oriented link with orientation induced from the orientation of (M, L) by q. If the reflection α is normal, then we say that the imitation

$$q : (M^*, L^*) \longrightarrow (M, L)$$

is a *normal imitation*. If α is isotopically almost standard, then we say that the imitation

$$q : (M^*, L^*) \longrightarrow (M, L)$$

is an *AID (=almost identical) imitation*. A normal imitation $q : (M^*, L^*) \longrightarrow (M, L)$ is said to be *imitation-homotopic* to a normal imitation $q' : (M^{*\prime}, L^{*\prime}) \longrightarrow (M, L)$ if there is an auto-homeomorphism h of $M \times I$ which is isotopic to the identity by an isotopy keeping $\partial(M \times I) \cup N(L) \times I$ fixed such that we have

$$q = \mathrm{proj}\phi : (M^*, L^*) \longrightarrow (M, L),$$
$$q' = \mathrm{proj}\phi' : (M^{*\prime}, L^{*\prime}) \longrightarrow (M, L)$$

for reflectors $\phi : (M^*, L^*) \longrightarrow (M, L) \times I$, $\phi' : (M^{*\prime}, L^{*\prime}) \longrightarrow (M, L) \times I$ of normal reflections $\alpha, h^{-1}\alpha h$ in $(M, L) \times I$, respectively. If $q : (M^*, L^*) \longrightarrow (M, L)$ is an AID imitation, then the restricted normal imitation

$$q|_{(M^*, L^* - K^*)} : (M^*, L^* - K^*) \longrightarrow (M, L - K)$$

for every component K of L and $K^* = q^{-1}(K)$, is imitation-homotopic to the identical imitation $1_{(M,L-K)} : (M, L-K) \longrightarrow (M, L-K)$. In particular, in the case of AID imitation, we can identify M^* with M and L^* with $(L - K) \cup K^*$ for every component K of L. From construction, we see that if $q^* : (M^{**}, L^{**}) \longrightarrow (M^*, L^*)$ and $q : (M^*, L^*) \longrightarrow (M, L)$ are normal (or AID, respectively) imitations, then there is a normal (or AID, respectively) imitation $q^{**} : (M^{**}, L^{**}) \longrightarrow (M, L)$ with $q^{**} = qq^*$ on a tubular neighborhood $N(L^{**})$ of L^{**} in M^{**}. The *exterior* of an oriented link (M, L) is the compact manifold $E(L) = \mathrm{cl}(M - N(L))$. A compact connected oriented 3-manifold M without boundary or with only torus boundary components is called a *hyperbolic* 3-manifold if $M - \partial M$ is a complete hyperbolic 3-manifold. Except for the hyperbolic 3-manifolds $S^1 \times D^2$ and $S^1 \times S^1 \times [0, 1]$, the hyperbolic 3-manifold M has a finite volume (see [18, C.7.2] for an explanation). Unless otherwise stated, hyperbolic 3-manifolds are assumed to have finite volomes. The volume and the isometry group of a hyperbolic 3-manifold M are denoted by $\mathrm{Vol}(M)$ and $\mathrm{Isom}(M)$ respectively, which are topological invariants of M by the Mostow rigidity theorem (see G. D. Mostow [21], W. P. Thurston [24, 25]). A hyperbolic 3-manifold M is said to be *asymmetric* if the isometry group $\mathrm{Isom}(M)$ is trivial. An imitation $q : (M^*, L^*) \longrightarrow (M, L)$ is called a *hyperbolic asymmetric imitation* if the exterior $E(L^*)$ is hyperbolic and asymmetric. The following lemma is proved in [8] except the asymmetry condition which is proven in [9].

Lemma 4.2. Let (M, L) be a disconnected oriented link. Then for any positive number C, there is a hyperbolic asymmetric AID imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

with $\mathrm{Vol}(E(L^*)) > C$.

From a technical reason, we need the following lemma:

Lemma 4.3. Let $q : M^* \longrightarrow M$ be an imitation such that M is a hyperbolic 3-manifold. Then there is a connected sum decomposition $M^* = S \# M'$ such that

(1) the connected summand S is a homology 3-sphere with $q_\#(\pi_1(S_0)) = \{1\}$ for the punctured manifold S_0 of S used for the connected sum,
(2) the connected summand M' is an irreducible 3-manifold and if ∂M^* is not empty, then it is a Haken manifold with incompressible boundary,
(3) the restriction $q|_{M'_0} : M'_0 \longrightarrow M$ of q to the compact punctured manifold M'_0 of M' used for the connected sum extends to a map $q' : (M', \partial M') \longrightarrow$

$(M, \partial M)$ whose lift

$$\tilde{q}' : (\tilde{M}', \partial\tilde{M}') \longrightarrow (\tilde{M}, \partial\tilde{M})$$

associated with every covering $\tilde{M}$ of M is a homology equivelence.

Proof. We use the homology equivalence property of imitation in [6], saying that the lift $\tilde{q} : \tilde{M}^* \longrightarrow \tilde{M}$ of the imitation map q associated with every covering $\tilde{M}$ of M induces isomorphisms

$$\tilde{q}_* : H_*(\tilde{M}^*; Z) \cong H_*(\tilde{M}; Z),$$
$$\tilde{q}_* : H_*(\tilde{M}^*, \partial\tilde{M}^*; Z) \cong H_*(\tilde{M}, \partial\tilde{M}; Z).$$

First, we show the following assertion:

(4.3.1) If there is a connected sum decomposition $M^* = M_1^* \# M_2^*$, then we have $q_\#(\pi_1(M_i^*)) = \{1\}$ for some i.

If there is a connected sum decomposition $M^* = M_1^* \# M_2^*$ with $q_\#(\pi_1(M_i^*)) \neq \{1\}$ for $i = 1$ and 2, then we consider the universal covering $\tilde{M}$ of M whose interior is homeomorphic to the 3-space. Let $\tilde{q} : \tilde{M}^* \longrightarrow \tilde{M}$ be the associated lifting of q. Let S^2 be a 2-sphere in $\tilde{M}^*$ lifting the 2-sphere defining the connected sum $M^* = M_1^* \# M_2^*$. By the homology equivalence property, we see that $\tilde{M}^*$ is connected and $H_1(\tilde{M}^*; Z) = 0$, so that the 2-sphere S^2 splits $\tilde{M}^*$ into two connected submanifolds X_i $(i = 1, 2)$. Using that $\pi_1(M)$ is a torsion-free group and hence $q_\#(\pi_1(M_i^*))$ is an infinite group for $i = 1, 2$, we see that X_i is not compact for $i = 1$ and 2. This implies that the 2-sphere S^2 represents a non-zero element of $H_2(\tilde{M}^*; Z)$, contradicting that $H_2(\tilde{M}^*; Z) = H_2(\tilde{M}; Z) = 0$. This proves (4.3.1).

By applying (4.3.1) and the homology equivalence property to a prime decomposition of M^* (cf. J. Hempel [3]), we can conclude that there is a connected sum decomposition $M^* = S \# M'$ such that S is a closed 3-manifold with $q_\#(\pi_1(S)) = \{1\}$ and M' is a prime 3-manifold. Since $q_\#(\pi_1(M')) = \pi_1(M)$ is a non-abelian hyperbolic group, we see that M' is an irreducible 3-manifold. For the universal covering space $\tilde{M}$ of M, we have that $H_1(\tilde{M}^*; Z) = 0$ and $\tilde{M}^*$ contains an infinitely many copies of S as connected summands. Thus, we have $H_1(S; Z) = 0$, showing that S is a homology 3-sphere, showing (1). If $\partial M'$ is not empty, then M' is a Haken manifold with incompressible boundary, because the restriction

$$q|_{\partial M^*} : \partial M^* = \partial M' \longrightarrow \partial M$$

of the imitation map q is a homotopy equivalence (see [6]) and ∂M is incompressible in M, showing (2) (cf. W. Jaco [4] for an account of Haken manifold and incompressibility). Let

$$W = M^* \times [0, 1] \cup_{j=1}^{s} h_j^2$$

be a cobordism from $M^* = M^* \times 0$ to M' such that h_j^2 $(j = 1, 2, \dots, s)$ are mutually disjoint 2-handles on the connected summand $S \times 1$ of $M^* \times 1$ whose surgery produce the connected summand S^3 of M'. Because $q_\#(\pi_1(S)) = \{1\}$, the map $q : M^* \longrightarrow M$ extends to a map $F : W \longrightarrow M$. Let $q' = F|_{M'} : M' \longrightarrow M$. For every covering $\tilde{M} \longrightarrow M$, we have a map

$$\tilde{F} : \tilde{W} \longrightarrow \tilde{M}$$

lifting F and extending the liftings $\tilde{q} : \tilde{M}^* \longrightarrow \tilde{M}$ and $\tilde{q}' : \tilde{M}' \longrightarrow \tilde{M}$ of q and q', respectively. By excision, we see that $H_d(\tilde{W}, \tilde{M}^*; Z) = H_d(\tilde{W}, \tilde{M}'; Z) = 0$ for $d \neq 2$. Using that S is a homology 3-sphere, we have natural isomorphisms

$$H_d(\tilde{M}^*; Z) \xrightarrow{\cong} H_d(\tilde{W}; Z) \xleftarrow{\cong} H_d(\tilde{M}'; Z)$$

for $d \neq 2$ and natural monomorphisms

$$H_2(\tilde{M}^*; Z) \longrightarrow H_2(\tilde{W}; Z) \quad \text{and} \quad H_2(\tilde{M}'; Z) \longrightarrow H_2(\tilde{W}; Z)$$

with the same image. Then the isomorphism $\tilde{q}_* : H_*(\tilde{M}^*; Z) \cong H_*(\tilde{M}; Z)$ induces an isomorphism $(\tilde{q}')_* : H_*(\tilde{M}'; Z) \cong H_*(\tilde{M}; Z)$. When $\partial M'$ is not empty, the restriction $\tilde{q}'|_{\partial M'} : \partial M' \longrightarrow \partial M$ is a homotopy equivalence, and hence by the five lemma we have an isomorphism

$$(\tilde{q}')_* : H_*(\tilde{M}', \partial \tilde{M}'; Z) \longrightarrow H_*(\tilde{M}, \partial \tilde{M}),$$

showing (3). □

5. Distinguished coloring of a framed link in a 3-manifold

In this section, we explain the basic result (Theorem 5.3) on changing the Dehn surgery description of a disconnected colored link. Its consequences are shown by assuming Theorem 5.3. We begin with the definitions of distinguished and π_1-distinguished colorings.

Definition 5.1. A coloring f of a framed link (M, L) is *distinguished* if we have $[\chi(L, f)] \neq [\chi(L, f')]$ for every coloring f' of L with $f' \neq f$, and π_1-*distinguished* if the fundamental groups $\pi_1(\chi(L, f))$ and $\pi_1(\chi(L, f'))$ are not isomorphic to each other for every coloring f' of L with $f' \neq f$.

The ∞-coloring of any framed link (M, L) with a trivial component is not distinguished. The ∞-coloring of every non-trivial knot in S^3 is distinguished by the Gordon-Luecke theorem [2], and π_1-distinguished if and only if the property P conjecture is true. If the distinguished ∞-coloring is the π_1-distinguished ∞-coloring for every oriented link in S^3 with ∞-coloring distinguished if and only if the Poincaré conjecture is true (see Remark 5.8 for the proof of the "only if"part). The following observation is important to our argument:

Lemma 5.2. The ∞-coloring of a framed link (M, L) is distinguished if and only if every homeomorphism $h : E(L') \longrightarrow E(L)$ from the exterior $E(L')$ of an oriented link (M, L') to the exterior $E(L)$ sends every meridian system $m(L')$ to a meridian system $m(L)$ up to orientations, so that h extends to a homeomorphism $h^+ : (M, L') \longrightarrow (M, L)$.

Proof. To prove the "only if"part, we consider an oriented link (M, L') with a homeomorphism $h : E(L') \longrightarrow E(L)$. Since the ∞-coloring of (M, L) is distinguished, the image $h(m(L'))$ must be equal to a meridian-system $m(L)$ in $E(L)$ up to orientations of $m(L)$. Hence we can extend h to a homeomorphism $h^+ : (M, L') \longrightarrow (M, L)$. To prove the "if"part, suppose that the ∞-coloring of a framed link (M, L) is not distinguished. Then there is a coloring $f \neq \infty$ of (M, L) such that $\chi(L, f) = M$, and the dual link of L in $\chi(L, f)$ is an oriented link (M, L') with $E(L') = E(L)$ such that the meridian system $m(L')$ of (M, L') up to orientations is not homologous to $m(L)$ in $\partial E(L)$. □

An oriented link (M, L) is *determined by the exterior* $E(L)$ *up to orientations* if we have a homeomorphism $(M, L') \cong (M, L)$ for every oriented link (M, L') with a homeomorphism $E(L') \cong E(L)$. By Lemma 5.2, every oriented link with the ∞-coloring distinguished is determined by the exterior up to orientations. A trivial knot and a Hopf link are examples of oriented links whose ∞-colorings are not distinguished. An oriented link (M, L) is *totally hyperbolic* if every non-empty sublink L_s is a hyperbolic link in M, that is, if the exterior $E(L_s)$ is a hyperbolic 3-manifold. If $q : (M, L^*) \longrightarrow (M, L)$ is a normal imitation of a framed link (M, L), then we can consider (M, L^*) as a framed link so that the imitation map q preserves meridian-longitude systems of L^* and L. Further, if $M = S^3$, then q preserves canonical meridian-longitude systems of L^* and L by the homology equivalence property in [6]. If f is a coloring of a framed link (M, L), then fq is a coloring of the framed link (M, L^*). Then we obtain the following theorem:

Theorem 5.3. Let (M, L) be a disconnected framed link. For every finite regular coloring f_L of L, any positive integer $\rho(f_L) \leq J$ and any positive number C, we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$,
(2) the Dehn surgery manifolds $\chi(L^*, fq)$ for all distinct colorings f of L with $\rho(f) \leq J$ and $f \neq \infty, f_L$ are mutually distinct hyperbolic asymmetric 3-manifold with volumes greater than C, and

(3) the Dehn surgery manifold $\chi(L^*, fq)$ for every coloring f of L with $\rho(f) \not\leq J$ is a normal imitation of a hyperbolic asymmetric 3-manifold with volume greater than C.

For a compact connected oriented 3-manifold M without boundary or with only torus boundary components, the Gromov norm $\|M\|$ is defined and is a constant multiple of the hyperbolic volume $\mathrm{Vol}(M)$ when M is a hyperbolic 3-manifold (see W. P. Thurston [24, 25]). The following theorem is a fundamental group version of Theorem 5.3:

Theorem 5.4. Let (M, L) be a disconnected framed link. For every finite regular coloring f_L of L, we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) L^* is totally hyperbolic,
(2) $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$, and
(3) the fundamental group $\pi_1(\chi(L^*, fq))$ for every coloring f of L with $f \neq \infty, f_L$ admits an epimorphism onto a non-abelian hyperbolic group and is not isomorphic to the fundamental group $\pi_1(\chi(L, f'))$ for every coloring f' of L.

Proof. In Theorem 5.3, we consider all colorings f of L such that $f(L) \subset \{\infty, \emptyset\}$ but $f \neq \infty$. Since $\rho(f) \leq 1 \leq J$, we see that L^* is totally hyperbolic. In Theorem 5.3, we consider the Dehn surgery manifold $M^* = \chi(L^*, fq)$ for every $f \neq \infty, f_L$ a normal imitation of a hyperbolic 3-manifold H with the imitation map $q_H : M^* \longrightarrow H$ such that the Gromov norm $\|H\| > C$, where we take $C \geq \|E(L)\|$. Then we have $C \geq \|\chi(L, f')\|$ for all colorings f' of L by a property of the Gromov norm (see W. P. Thurston [24, 25]). Suppose that $\pi_1(M^*)$ is isomorphic to the fundamental group $\pi_1(N)$ of the Dehn surgery manifold $N = \chi(L, f')$ for some coloring f' of L. By Lemma 4.3, there is a connected sum $M^* = S \# M'$ such that S is a homology 3-sphere and M' is an irreducible manifold with a degree one map $q'_H : (M', \partial M') \longrightarrow (H, \partial H)$. By a property of the Gromov norm (see W. P. Thurston [24, 25]), we have $\|M'\| \geq \|H\| > C$. Since ∂N has only torus components and every compact oriented 3-manifold with a positive genus boundary component has a non-trivial first homology, we see from Kneser's conjecture (see J. Hempel [3]) that there is a connected sum $N = S' \# N'$ such that S' is a homology 3-sphere and N' is an irreducible 3-manifold homotopy equivalent to M'. Then we show that

$$\|N'\| = \|M'\|.$$

To see this, first, assume that ∂N is empty. Then $\partial M' = \partial M$ is empty and we have degree one maps $N' \longrightarrow M'$ and $M' \longrightarrow N'$, so that $\|N'\| = \|M'\|$ by a property of the Gromov norm. Next, assume that ∂N is not empty. Then M' and N' are Haken manifolds with incompressible boundary consisting of torus components. By the Johannson theorem (see W. Jaco [4, p.212]), the hyperbolic pieces of the torus decompositions of N' and M' are mutually homeomorphic. By T. Soma [23], $\|N'\|$ and $\|M'\|$ are equal to the sums of the Gromov norms of the hyperbolic pieces of the torus decompositions of N' and M', respectively. Hence we have $\|N'\| = \|M'\|$ as desired.

Since there is a degree one map $N \longrightarrow N'$, we have

$$\|N\| \geq \|N'\| = \|M'\| > C$$

by a property of the Gromov norm, which contradicts $C \geq \|N\|$. Thus, we see that $\pi_1(\chi(L^*, fq)) = \pi_1(M^*)$ is not isomorphic to $\pi_1(N) = \pi_1(\chi(L, f'))$. □

We see from (3) of Theorem 5.4 that the link L^* is distinct from the link L componentwise, by considering collorings f, f' of L such that

$$f(L - K) = f'(L - K') = \{\infty\} \quad \text{and} \quad f(K) = f'(K') \in \mathbb{Q}$$

for every pair of components K, K' of L. The existence of this kind of imitation on the Dehn surgery of a colored disconnected link in S^3 has been promised in [7, p.151]. Counting this observation, we obtain the following three corollaries from Theorem 5.4:

Corollary 5.5. Let (M, L) be a disconnected framed link, and f_L a finite regular coloring of L such that $[\chi(L, f_L)] = [M]$. Then we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) L^* is totally hyperbolic and distinct from L componentwise,
(2) $[\chi(L^*, f_L q)] = [\chi(L, f_L)] = [\chi(L^*, \infty)] = [M]$, and
(3) the fundamental group $\pi_1(\chi(L^*, fq))$ for every coloring f of L with $f \neq \infty, f_L$ is not isomorphic to the fundamental group $\pi_1(\chi(L, f'))$ for every coloring f' of L.

Corollary 5.6. Let (M, L) be a disconnected framed link. Assume that the fundamental group $\pi_1(\chi(L, f_L))$ is not isomorphic to the fundamental group $\pi_1(M)$ for a finite regular coloring f_L of L. Then we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) L^* is totally hyperbolic and distinct from L componentwise,
(2) $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$, and
(3) the colorings ∞ and $f_L q$ of L^* are π_1-distinguished.

Corollary 5.7. Assume that a disconnected framed link (M, L) has $[\chi(L, f_L)] \neq [M]$ for a finite regular coloring f_L of L. Then we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) L^* is totally hyperbolic and distinct from L componentwise,
(2) $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$, and
(3) the colorings ∞ and $f_L q$ of L^* are distinguished.

Concerning Corollary 5.7, here is a remark.

Remark 5.8. For every $[M] \in \mathbb{M}$ with $[M] \neq [S^3]$, we see from W. B. R. Lickorish [20], A. H. Wallace [26] and Corollary 5.7 that we have a colored link type $[L, f] \in \mathbb{L}^{\mathrm{f}}$ such that $[\chi(L, f)] = [M]$ and the ∞-coloring of L is distinguished. If the distinguished ∞-coloring is the π_1-distinguished ∞-coloring for every oriented link in S^3 with ∞-coloring distinguished, then the fundamental group $\pi_1(M)$ is non-trivial and hence the Poincaré conjecture is affirmative.

The following example leads to a hyperbolic group classification of closed connected oriented 3-manifolds:

Example 5.9. We shall construct an injective map

$$a^0_\infty : \mathbb{M} \longrightarrow \mathbb{L}^0$$

which is a faithful right inverse of D^0, analogous to but different from a^0 in Example 3.4. For every $x \in \mathbb{M}$, we take the subset $\mathbb{L}^0_\infty(x) \subset \mathbb{L}^0$ consisting of a 0-colored link type $x^0 = [L, 0]$ such that L is an oriented hyperbolic link *possibly with infinite volume* determined by the exterior $E(L)$ up to orientations, and $D^0(x^0) = [\chi(L, 0)] = x$. The set $\mathbb{L}^0_\infty(x)$ is not empty for every $x \in \mathbb{M}$. In fact, if $x \neq [S^3]$, then we see from Lemma 2.3 and Corollary 5.7 that $\mathbb{L}^0_\infty(x)$ is not empty. If $x = [S^3]$, then the set $\mathbb{L}^0_\infty(x)$ is also non-empty, since it contains a 0-colored Hopf link type. Since $D^0\mathbb{L}^0_\infty(x) = \{x\}$ and $\mathbb{L}^0_\infty(\bar{x}) = \overline{\mathbb{L}^0_\infty(x)}$ for every $x \in \mathbb{M}$, we obtain from Criterion 3.2 a unique faithful right inverse

$$a^0_\infty : \mathbb{M} \longrightarrow \mathbb{L}^0$$

of the Dehn surgery map D^0 with $a^0_\infty(x) \in \mathbb{L}^0_\infty(x)$ for every $x \in \mathbb{M}$ such that $a^0_\infty(x) = \psi\mathbb{L}^0_\infty(x)$ if $x = \bar{x}$, and either $a^0_\infty(x)$ or $a^0_\infty(\bar{x})$ is equal to $\psi(\mathbb{L}^0_\infty(x) \cup \mathbb{L}^0_\infty(\bar{x}))$ if $x \neq \bar{x}$. By the natural identification $\mathbb{L}^0 = \mathbb{L}$, the map a^0_∞ is identified with the injection

$$a_\infty : \mathbb{M} \longrightarrow \mathbb{L}$$

such that we have $a_\infty(\bar{x}) = \overline{a_\infty(x)}$ for every $x \in \mathbb{M}$ with $x \neq \bar{x}$. We note that $a_\infty([S^3])$ is the positive Hopf link type and $a_\infty([S^1 \times S^2])$ is the trivial knot type.

By making use of Example 5.9, we show the following result:

Theorem 5.10. To every closed connected oriented 3-manifold M we can assign a finitely generated torsion-free discrete subgroup G_M of $PSL(2, \mathbb{C})$ so that there is a homeomorphism $M \longrightarrow M'$ if and only if there is an isomorphism $G_M \longrightarrow G_{M'}$. The group G_M is abelian if and only if M is homeomorphic to S^3 or $S^1 \times S^2$ where we have $G_{S^3} \cong Z \oplus Z$ and $G_{S^1 \times S^2} \cong Z$.

Proof. Let $a_\infty([M]) = [L_M]$ in Example 5.9. Let $G_M = \pi_1(E(L_M))$ be the group of a hyperbolic link L_M possibly with infinite volume, which is a finitely generated torsion-free discrete subgroup of $PSL(2, \mathbb{C})$. From our construction, we see that if there is a homeomorphism $M \longrightarrow M'$, then $[L_M] = [L_{M'}]$ or $[L_M] = [-\bar{L}_{M'}]$ according to whether $[M] = [M']$ or $[M] \neq [M']$ and $[M] = [-M']$. Hence there is an isomorphism $G_M \longrightarrow G_{M'}$. Conversely, assume that there is an isomorphism $G_M \longrightarrow G_{M'}$. A hyperbolic link has the infinite volume if and only if it is a trivial knot or a Hopf link, whose link group constitutes all the abelian link groups. If one of the hyperbolic manifolds $E(L_M)$ and $E(L_{M'})$ has a finite volume, then both of the hyperbolic manifolds have finite volumes and hence there is a homeomorphism $E(L_M) \longrightarrow E(L_{M'})$ by the Mostow rigidity theorem (see G. D. Mostow [21], W. P. Thurston [24, 25]). If both the hyperbolic manifolds $E(L_M)$ and $E(L_{M'})$ have the infinite volume, then L_M and L'_M are a trivial knot and/or an oriented Hopf link, so that the isomorphism $G_M \cong G_{M'}$ implies that there is a homeomorphism $E(L_M) \longrightarrow E(L_{M'})$. Since the oriented links $L_M, L_{M'}$ are determined by the exteriors up to orientations, we have a homeomorphism $(S^3, L_M) \longrightarrow (S^3, L_{M'})$, which implies the identity $[M] = [\pm M']$, namely there is a homeomorphism $M \longrightarrow M'$. In Example 5.9, we have $G_{S^3} \cong Z \oplus Z$ and $G_{S^1 \times S^2} \cong Z$. If $[M] \neq [S^3], [S^1 \times S^2]$, then the link L_M is a hyperbolic link which is neither a trivial knot nor a Hopf link, so that G_M is not ablelian. □

6. Proof of Theorem 5.3

We note that if (M, L) is a framed link and $q : (M, L^*) \longrightarrow (M, L)$ is a normal imitation, then L^* is a framed link by a meridian-longitude system induced from that of L by q, so that a colored link (L, f) induces a unique colored link (L^*, fq). The following lemma is obtained by combining Lemma 4.2 with the idea of [15, Lemma 2.1]:

Lemma 6.1. For any disconnected framed link (M, L), any positive number C and any positive integer J, there is an AID imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) $\chi(L^*, fq)$ is a hyperbolic asymmetric 3-manifold with volume greater than C for every finite coloring f of L with $\rho(f) \leq J$,
(2) $\chi(L^*, fq)$ and $\chi(L^*, f'q)$ are distinct, i.e., $[\chi(L^*, fq)] \neq [\pm\chi(L^*, f'q)]$, for every pair of distinct finite colorings f, f' of L with $\rho(f), \rho(f') \leq J$.

Proof. When $M = S^3$, the proof is proved in [15, Lemma 2.1] except the volume condition which can be easily added in the topological imitation theory. Since the present proof is parallel to the argument of [15, Lemma 2.1], we give here only the outline of the proof. Let L^+ be a meridian addition link of L, that is a link obtained from L by adding a meridian loop to every component of L. We note that the sublink $L^+ - L$ is canonically framed by which we consider L^+ a framed link extending the framed link L. By Lemma 6.1 we have a hyperbolic asymmetric AID imitation

$$q^+ : (M, (L^+)^*) \longrightarrow (M, L^+)$$

with $\mathrm{Vol}(E(L^+)^*) > C$ for every given positive number C. For any finite coloring f of L and a positive integer n, let f^n be the finite coloring of L^+ such that

$$f^n(K) = \begin{cases} f(K) + n & (\text{if } K \subset L) \\ \frac{1}{n} & (\text{if } K \subset L^+ - L). \end{cases}$$

We note that $\chi(L^+, f^n) = \chi(L, f)$ and $\chi(L^+ - L, \frac{1}{n}) = M$. Since

$$\lim_{n \to +\infty} \rho(f^n(K)) = +\infty$$

for every component K of L^+ and there are only finitely many colorings f of L with $\rho(f) \leq J$, we see from Thurston's hyperbolic Dehn surgery argument ([24, 25]) that if we take n sufficiently large, then the AID imitation

$$q = \chi(q^+; (L^+ - L, \frac{1}{n})) : (M, L^*) \longrightarrow (M, L)$$

obtained by taking the Dehn surgery manifold $\chi(L^+ - L, \frac{1}{n}) = M$ has the property that for every finite coloring f of L with $\rho(f) \leq J$ the Dehn surgery manifold $\chi(L^*, fq)$ is a hyperbolic asymmetric 3-manifold with volume greater than C and the dual link in $\chi(L^*, fq)$ of the sublink obtained from L^* by removing the sublink $L^*_\emptyset \subset L^*$ consisting of a component K^* with $fq(K^*) = \emptyset$ consists of short geodesics. This last condition together with the Mostow rigidity theorem ([21,24,25]) implies that $[\chi(L^*, fq)] \neq [\pm\chi(L^*, f'q)]$ for every pair of distinct colorings f, f' of L with $\rho(f), \rho(f') \leq J$ (see [15, Lemma 2.1]). □

In Lemma 6.1, if f is an infinite coloring of L, then we have $[\chi(L^*, fq)] = [\chi(L, f)]$ by a property of the AID imitation q. We used first this property in an argument of Dehn surgery of [8, Corollary 4.1], which is also developed in the following lemma:

Lemma 6.2. Let L_s and $L_s^c = L - L_s$ be non-empty sublinks of a framed link (M, L). For any positive number C, any positive integer J and any finite regular coloring f_L of L, we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

where L^* is written as $L_s^* \cup L_s^c$ such that

(1) the restriction

$$q|_{(M, L^* - K^*)} : (M, L^* - K^*) \longrightarrow (M, L - K)$$

for every knot $K^* \subset L_s^*$ and the knot $K = q(K^*) \subset L_s$ is imitation-homotopic to the identical imitation,

(2) $[\chi(L^*, fq)] = [\chi(L, f)]$ for every coloring f of L such that $f(K) = f_L(K)$ for a knot $K \subset L_s^c$,

(3) the Dehn surgery manifolds $\chi(L^*, fq)$ for all distinct colorings f of L such that $\rho(f) \leq J$, $f|_{L_s}$ is a finite coloring of L_s, and $f(K) \neq f_L(K)$ for any knot $K \subset L_s^c$ are mutually distinct hyperbolic asymmetric 3-manifolds with volumes greater than C.

Proof. We consider the Dehn surgery manifold $M' = \chi(L_s^c, f_L|_{L_s^c})$ and the framed link $L' = L_s \cup L_s'^c$ in M' where $L_s^{c\prime}$ denotes the dual framed link obtained from the link L_s^c by the Dehn surgery operation $M \longrightarrow M'$. We apply Lemma 6.1 to (M', L') to obtain an AID imitation

$$q' : (M', (L')^*) \longrightarrow (M', L')$$

where $(L')^*$ can be written as $L_s^* \cup L_s^{c\prime}$. By the dual Dehn surgery operation $M' \longrightarrow M$, the AID imitation q' induces a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

where L^* can be written as $L_s^* \cup L_s^c$. (1) follows directly, since the normal imitation

$$q'|_{(M', (L')^* - K^*)} : (M, L^* - K^*) \longrightarrow (M', L' - K)$$

is imitation-homotopic to the identical imitation. Since the coloring f of L in (2) changes into an infinite coloring f' of the framed link (M', L'), we obtain (2) from the remark preceding to this lemma. Since the coloring f of L in (3) changes into a finite coloring f' of the framed link (M', L'), (3) follows from the properties of Lemma 6.1 with a large positive integer J. $\square$

An important observation on Lemma 6.2 is that the coloring f of (3) may be infinite on L_s^c.

Lemma 6.3. Let (M, L) be a disconnected framed link. For any positive number C, any positive integer J and any finite regular coloring f_L of L, we have a normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that

(1) $[\chi(L^*, f_L q)] = [\chi(L, f_L)]$,

(2) the Dehn surgery manifolds $\chi(L^*, fq)$ for all distinct colorings f of L such that $\rho(f) \leq J$ and $f \neq \infty, f_L$ are mutually distinct hyperbolic asymmetric 3-manifolds with volumes greater than C.

Proof. Let L_i $(i = 1, 2, \ldots, m)$ be all the non-empty sublinks of L such that $L_i^c = L - L_i$ is not empty for all i. Inductively, we take positive numbers C_i $(i = 1, 2, \ldots, m)$ and m normal imitations

$$q_i : (M, L_i^* \cup L_i^{c*}) \longrightarrow (M, L_{i-1}^* \cup L_{i-1}^{c*}) \quad (i = 1, 2, \ldots, m)$$

which satisfy the following conditions:

(i) $L_0^* = L_1$, $L_0^{c*} = L_1^c$, $C_1 = C$.

(ii) When we regard L_{i-1}^*, L_{i-1}^{c*} and C_i as L_s, L_s^c and C in Lemma 6.2 respectively, we take q and L^* in Lemma 6.2 as q_i and $L_i^* \cup L_i^{c*}$ where we take L_i^* and L_i^{c*} so that

$$q_i q_{i-1} \ldots q_1(L_i^*) = L_i, \quad q_i q_{i-1} \ldots q_1(L_i^{c*}) = L_i^c.$$

(iii) $\|E(L_i^* \cup L_i^{c*})\| > C_i \geq \|E(L_{i-1}^* \cup L_{i-1}^{c*})\| \quad (i = 1, 2, \ldots, m)$.

Taking $L^* = L_m^* \cup L_m^{c*}$, we have a composite normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that $q = q_m q_{m-1} \ldots q_1$ on a tubular neighborhood $N(L^*)$ of L^* in M. We show that this normal imitation q has the properties (1) and (2). (1) follows directly from (2) of Lemma 6.2. To see (2), let f be a coloring of L such that $\rho(f) \leq J$ and $f \neq \infty, f_L$. Let $L_{f=f_L}$ be the sublink of L consisting of a component K of L with $f(K) = f_L(K)$, and $L_{f=\infty}$ the sublink of L consisting of a component K of L with $f(K) = \infty$. By the assumption that $f \neq \infty, f_L$, the sublinks $L_{f=f_L}$ and $L_{f=\infty}$ are disjoint proper sublinks of L (which may be empty). We take the largest index i such that $L_{f=f_L} \subset L_i$ and $L_{f=\infty} \subset L_i^c$. By (1) and (2) of Lemma 6.2, we have

$$[\chi(L^*, fq)] = [\chi(L_i^* \cup L_i^{c*}, f q_i q_{i-1} \ldots q_1)].$$

By (3) of Lemma 6.2, the Dehn surgery manifolds $\chi(L_i^* \cup L_i^{c*}, fq_iq_{i-1}\dots q_1)$ for all distinct colorings f with $\rho(f) \le J$ and $f \ne \infty, f_L$ such that $L_{f=f_L} \subset L_i$ and $L_{f=\infty} \subset L_i^c$ are mutually distinct hyperbolic asymmetric 3-manifolds with volumes greater than C_i. Since the volumes of these hyperbolic 3-manifolds is smaller than or equal to C_{i+1}, we see (2). □

Proof of Theorem 5.3. For $J_1 = J$, by Lemma 6.3 we have a normal imitation

$$q^1 : (M, L^{*1}) \longrightarrow (M, L)$$

such that $[\chi(L^{*1}, f_L q^1)] = [\chi(L, f_L)]$ and the Dehn surgery manifolds $\chi(L^{*1}, fq^1)$ for all distinct colorings f of L such that $\rho(f) \le J_1$ and $f \ne \infty, f_L$ are mutually distinct hyperbolic asymmetric 3-manifolds with volumes greater than C. Then by Thurston's argument on hyperbolic Dehn surgery, there exists an integer $J_1^+ > J_1$ such that

(*) the Dehn surgery manifolds $\chi(L^{*1}, fq^1)$ are mutually distinct hyperbolic 3-manifolds with volumes greater than C for all distinct colorings f of L such that $f \ne \infty, f_L,\ \rho(f|_{L_s}) \le J_1$ and $\rho(f|_{L-L_s}) > J_1^+$ for a (possibly empty) sublink $L_s \subset L$.

Let $J_2 = J_1^+$. Let L have the r components K_i $(i = 1, 2, \dots, r)$. Then by continuing this process, there are integers $J_j (j = 1, 2, \dots, r + 2)$ with $J_{r+2} > J_{r+1} > \cdots > J_1 = J$ and normal imitations

$$q^j : (S^3, L^{*j}) \longrightarrow (S^3, L^{*(j-1)}) \quad (j = 1, 2, \dots, r + 1)$$

where $L^{*0} = L$ such that

$$[\chi(L^*, f_L q^{r+1} q^r \dots q^1)] = [\chi(L, f_L)]$$

and we have the following condition for every $j = 1, 2, \dots, r + 1$:

(**) The Dehn surgery manifolds $\chi(L^{*j}, fq^j q^{j-1} \dots q^1)$ are mutually distinct hyperbolic 3-manifolds with volumes greater than C for all distinct colorings f of L such that $f \ne \infty, f_L,\ \rho(f|_{L_s}) \le J_j$ and $\rho(f|_{L-L_s}) > J_{j+1}$ for a (possibly empty) sublink $L_s \subset L$.

Since the component number of L is r, for every coloring f of L we can find an index j such that none of the sizes $\rho(f(K_i))$ for all i are in the half open interval $(J_j, J_{j+1}]$, so that every coloring f of L with $f \ne \infty, f_L$ satisfies the condition in (**) for some j and hence the Dehn surgery manifold $\chi(L^{*j}, fq^j q^{j-1} \dots q^1)$ is a hyperbolic 3-manifold with volume greater than C. Taking $L^* = L^{*(r+1)}$, we have a composite normal imitation

$$q : (M, L^*) \longrightarrow (M, L)$$

such that $q = q^{r+1}q^r \dots q^1$ on a tubular neighborhood $N(L^*)$ of L^* in M. Since the Dehn surgery manifold $\chi(L^*, fq)$ is a normal imitation of the Dehn surgery manifold $\chi(L^{*j}, fq^j q^{j-1} \dots q^1)$ for every coloring f and every j, the normal imitation q is a desired imitation with J_r as J. $\square$

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