On certain maximal cyclic modules for the quantized special linear algebra at a root of unity *

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Abstract

By properly specializing the parameters irreducible modules of maximal dimension for the De Concini-Kac version of the Drinfeld-Jimbo quantum algebra in type A may be transformed into modules over Lusztig's infinitesimal quantum algeba. Thus obtained modules have a simple socle and a simple head, and share the same dimension as the infinitesimal Verma modules. Despite these common features we find that they are never isomorphic to infinitesimal Verma modules unless they are irreducible. The same carry over to the modular setup for the special linear groups in positive characteristic.

The finite dimensional irreducible representations of the De Concini-Kac version of the Drinfeld-Jimbo quantized enveloping algebra at a complex ℓ-th root of 1 have their dimensions bounded above, generically attaining a maximal dimension [DK]. In type A Date, Jimbo, Miki and Miwa [DJMM] have given a concrete realization of most of those of maximal dimension. By properly specializing their parametres the second named author of the present paper found in [N] that they afford modules $\mathcal{V}$, rarely irreducible, for Lusztig's "infinitesimal" quantum algebra $\mathfrak{u}$ [L1] and that each $\mathcal{V}$ has a unique, up to scalar, invariant vector $u_{\bar{0}}$ relative to a Borel subalgebra $\mathfrak{u}^\sharp$ of $\mathfrak{u}$, and hence that $\mathcal{V}$ has a simple socle generated by $u_{\bar{0}}$. The dimension being right, it is tempting to compare $\mathcal{V}$ with Humphreys' "infinitesimal" Verma modules [H] quantized by Andersen, Polo and Wen [APW], which are the standard objects of study in the representation theory of $\mathfrak{u}$.

It is easy to see that $\mathcal{V}$ is isomorphic to an infinitesimal Verma module as $\mathfrak{u}^\sharp$-module, which in turn shows that $\mathcal{V}$ has the same simple head as the infinitesimal Verma module. The explicit description of the actions of the standard generators of $\mathfrak{u}$ on $\mathcal{V}$ allows us, however, to find that $\mathcal{V}$ has also a unique, up to scalar, invariant vector u_- with respect to the opposite infinitesimal Borel subalgebra. It follows that $\mathcal{V}$ does not lift to an integrable $\mathfrak{u}U^0_{\mathbb{C}}$-module, and hence $\mathcal{V}$ can not be isomorphic to any infinitesimal Verma module as $\mathfrak{u}$-module unless $\mathcal{V}$ is simple, where $U^0_{\mathbb{C}}$ is the Cartan part of Lusztig's quantum algebra $U_{\mathbb{C}}$ at the ℓ-th root of 1.

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By construction $\mathcal{V}$ may be defined over $\mathcal{B} = \mathbb{Z}[v, v^{-1}]/(\phi_\ell)$ with ϕ_ℓ the ℓ-th cyclotomic polynomial in indeterminate v. If ℓ is an odd prime p, $\mathcal{B}/(v-1)$ is a finite field $\mathbb{F}_p$ of p-elements. Let G be the special linear group scheme over $\mathbb{F}_p$ with opposite Borel subgroups B and B^+, and let G_1, B_1, B_1^+ be the Frobenius kernel of G, B, and B_1^+, respectively. If $\mathcal{V}_\mathcal{B}$ is the $\mathcal{B}$-form of $\mathcal{V}$, then $\mathcal{V}_p = \mathcal{V}_\mathcal{B} \otimes_\mathcal{B} \mathbb{F}_p$ is naturally a G_1-module. We find that $u_{\bar{0}} \otimes 1$ (resp. $u_- \otimes 1$) remains a unique, up to $\mathbb{F}_p^\times$, B_1^+- (resp. B_1-)invariant vector in $\mathcal{V}_p$. Hence $\mathcal{V}_p$ is isomorphic to an infinitesimal Verma module as B_1^+-module, but not as G_1-module unless $\mathcal{V}_p$ is simple.

If the simple $\mathfrak{u}$-module generated by $u_{\bar{0}}$ in $\mathcal{V}$ and the simple G_1-module generated by $u_{\bar{0}} \otimes 1$ in $\mathcal{V}_p$ have the same dimension, Lusztig's conjecture for the irreducible characters of G-modules will follow from the celebrated theorems of Kazhdan and Lusztig [KL] and Kashiwara and Tanisaki [KT] and Casian [C].

If $\mathcal{C}$ is a category, $\mathcal{C}(A, B)$ will denote the set of morphisms of $\mathcal{C}$ from object X of $\mathcal{C}$ to object Y of $\mathcal{C}$. If A is a ring, A**Mod** will denote the category of left A-modules.

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1° Infinitesimal Verma modules

In this section we recollect some facts about infinitesimal Verma modules over an arbitrary quantum algebra of finite type.

(1.1) Let $\mathbb{Q}(v)$ be the fractional field of the Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ in indeterminate v, $A = [\![A_{ij}]\!]$ an indecomposable Cartan matrix of finite type and let $\mathbf{U}$ be the associated Drinfeld-Jimbo quantum algebra over $\mathbb{Q}(v)$ with generators E_i, F_i, and $K_i^{\pm 1}$, $i \in [1, n]$. Let U be Lusztig's $\mathcal{A}$-subalgebra of $\mathbf{U}$ generated by $E_i^{(r)} = \frac{E_i^r}{[r]_i!}$, $F_i^{(r)} = \frac{F_i^r}{[r]_i!}$, $K_i^{\pm 1}$, $i \in [1, n]$, $r \in \mathbb{N}$, where $[r]_i! = \prod_{s=1}^r [s]_i$ with $[s]_i = \frac{v_i^s - v_i^{-s}}{v_i - v_i^{-1}}$, $v_i = v^{d_i}$, $d_i \in \{1, 2, 3\}$ minimal such that the matrix $[\![d_i A_{ij}]\!]$ is symmetric.

Let R (resp. Λ) be the root system (resp. the weight lattice) associated to A and R^+ a positive subsystem of R with the simple roots α_i, $i \in [1, n]$. We equip Λ with a partial order defined by R^+ as usual. Let $\Lambda^\vee$ be the colattice of Λ and denote by $\langle\ ,\ \rangle : \Lambda \times \Lambda^\vee \to \mathbb{Z}$ the perfect pairing. If $\alpha \in R$, let $\alpha^\vee$ be its coroot. We set $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$ for $\lambda \in \Lambda$. Let U^0 be the $\mathcal{A}$-subalgebra of U generated by $K_i^{\pm 1}$ and $\begin{bmatrix} K_i; c \\ r \end{bmatrix} = \prod_{s=1}^r \frac{K_i v_i^{c-s+1} - K_i^{-1} v_i^{-c+s-1}}{v_i^s - v_i^{-s}}$, $i \in [1, n]$, $c \in \mathbb{Z}$, $r \in \mathbb{N}$. Each $\lambda \in \Lambda$ defines an $\mathcal{A}$-algebra homomorphism $\chi_\lambda : U^0 \to \mathcal{A}$ such that

$$K_i \mapsto v_i^{\lambda_i}, \quad \begin{bmatrix} K_i; c \\ r \end{bmatrix} \mapsto \begin{bmatrix} \lambda_i + c \\ r \end{bmatrix}_i = \frac{\prod_{s=0}^{r-1} [\lambda_i + c - s]_i}{[r]_i!} \quad \forall i \in [1, n], c \in \mathbb{Z}, r \in \mathbb{N}.$$

(1.2) Let ℓ be a positive integer greater than 2 prime to all entries A_{ij} of the Cartan matrix A, $\mathcal{K} = \mathbb{Q}[v]/(\phi_\ell)$, and set $U_\mathcal{K} = U \otimes_\mathcal{A} \mathcal{K}$. Let $\mathfrak{u}$ (resp. $\mathfrak{u}^+$; $\mathfrak{u}^-$; $\mathfrak{u}^0$) be the $\mathcal{K}$-subalgebra of $U_{\mathcal{K}}$ generated by $E_i \otimes 1$, $F_i \otimes 1$, $K_i \otimes 1$ (resp. $E_i \otimes 1$; $F_i \otimes 1$; $K_i \otimes 1$), $i \in [1,n]$. Let also $\mathfrak{u}^\sharp = \mathfrak{u}^+\mathfrak{u}^0$ and $\mathfrak{u}^\flat = \mathfrak{u}^0\mathfrak{u}^-$. We will abbreviate $x \otimes 1$ of $U_{\mathcal{K}}$ as x, and $\chi_\lambda \otimes_{\mathcal{A}} \mathcal{K}$ as χ_λ.

Let $\tilde{\mathfrak{u}}$ be the $\mathcal{K}$-subalgebra of $U_{\mathcal{K}}$ generated by $\mathfrak{u}$ and $U^0_{\mathcal{K}} = U^0 \otimes_{\mathcal{A}} \mathcal{K}$, and let $\tilde{\mathfrak{u}}^\sharp = \mathfrak{u}^\sharp U^0_{\mathcal{K}}$, $\tilde{\mathfrak{u}}^\flat = U^0_{\mathcal{K}}\mathfrak{u}^\flat$. Each $\lambda \in \Lambda$ defines a 1-dimensional $\tilde{\mathfrak{u}}^\flat$-module by χ_λ annihilating all F_i, which we will still denote by λ. Let $\tilde{\nabla}(\lambda) = {}_{\tilde{\mathfrak{u}}^\flat}\mathbf{Mod}(\tilde{\mathfrak{u}}, \lambda)$. We make $\tilde{\nabla}(\lambda)$ into a $\tilde{\mathfrak{u}}$-module by setting $xf = f(?x)$ for each $x \in \tilde{\mathfrak{u}}$ and $f \in \tilde{\nabla}(\lambda)$.

Let $\Lambda^{\mathrm{res}} = \{\nu \in \Lambda \mid \nu_i \in [0, \ell - 1]\ \forall i\}$. If we write $\lambda = \lambda^0 + \ell\lambda^1$ with $\lambda^0 \in \Lambda^{\mathrm{res}}$ and $\lambda^1 \in \Lambda$, one has from [APW, 1.9] an isomorphism of $\tilde{\mathfrak{u}}$-modules

(1) $$\tilde{\nabla}(\lambda) \simeq \tilde{\nabla}(\lambda^0) \otimes_{\mathcal{K}} \ell\lambda^1,$$

where $\ell\lambda^1$ ia a 1-dimensional $\tilde{\mathfrak{u}}$-module defined by $\chi_{\ell\lambda^1}$ annihilating all E_i, F_i and $K_i - 1$.

On the other hand, the natural gradation on $\mathfrak{u}^+$ assigning each E_i grade α_i equips $\mathfrak{u}^+$ with a structure of $\tilde{\mathfrak{u}}^\sharp$-module such that $\mathfrak{u}^+$ act by the left multiplication and $U^0_{\mathcal{K}}$ by $\chi_{-\lambda+\nu}$ on the ν-th homogeneous part of $\mathfrak{u}^+$, $\nu \in \sum_i \mathbb{N}\alpha_i$. Recall antiautomorphism Ψ on U such that $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_i \mapsto K_i^{-1}$, $i \in [1,n]$. If M is a $\tilde{\mathfrak{u}}^\sharp$-module of finite type, we will denote by M^Ψ the $\mathcal{K}$-linear dual of M made into $\tilde{\mathfrak{u}}^\sharp$-module by setting $xf = f(\Psi(x)?)$ for each $x \in \tilde{\mathfrak{u}}^\sharp$, $f \in \mathbf{Mod}_{\mathcal{K}}(M, \mathcal{K})$. Then we have an isomorphism of $\tilde{\mathfrak{u}}^\sharp$-modules

(2) $$\tilde{\nabla}(\lambda) \simeq (\mathfrak{u}^+)^\Psi \quad \text{via} \quad f \mapsto f \circ (\Psi \otimes_{\mathcal{A}} \mathcal{K}).$$

One can likewise define a $\mathfrak{u}$-module $\nabla(\lambda) = {}_{\mathfrak{u}^\flat}\mathbf{Mod}(\mathfrak{u}, \lambda)$. By restricting the $\tilde{\mathfrak{u}}$-action to $\mathfrak{u}$, $\tilde{\nabla}(\lambda)$ yields $\nabla(\lambda)$. Then the isomorphism (2) restricts to an isomorphism of $\mathfrak{u}^\sharp$-modules

(3) $$\nabla(\lambda) \simeq (\mathfrak{u}^+)^\Psi.$$

(1.3) Recall from Xi [X, 2.5] that

(1) $$\mathfrak{u}^+ \text{ has a simple socle } \mathcal{K} \prod_{\alpha \in R^+} E_\alpha^{\ell-1} \text{ as } \mathfrak{u}^+\text{-module},$$

where E_α is a root vector of $\mathfrak{u}^+$ associated to $\alpha \in R^+$ [L2] and the product is taken in a certain specific order. It follows that $\mathfrak{u}^+$ is indecomposable as $\mathfrak{u}^+$-module, and hence that

(2) $$\mathfrak{u}^+ \text{ is a projective cover of trivial module } \mathcal{K} \text{ as } \mathfrak{u}^+\text{-module}.$$

By an integrable $\tilde{\mathfrak{u}}$- (resp. $\mathfrak{u}$-) module M we will mean a $\tilde{\mathfrak{u}}$- (resp. $\mathfrak{u}$-) module M such that

$$M = \coprod_{\mu \in \Lambda} M_\mu, \quad \text{with} \quad M_\mu = \{m \in M \mid tm = \chi_\mu(t)m\ \forall t \in U^0_{\mathcal{K}} \text{ (resp. } \mathfrak{u}^0)\}.$$

Each $\tilde{\nabla}(\lambda)$ (resp. $\nabla(\lambda)$) is an integrable $\tilde{\mathfrak{u}}$- (resp. $\mathfrak{u}$-) module. Define integrable $\tilde{\mathfrak{u}}^\sharp$- and $\mathfrak{u}^\sharp$-modules likewise. One obtains from (2) and (1.2.2, 3)

Proposition. *For each $\lambda \in \Lambda$ the $\mathfrak{u}^\sharp$- (resp. $\tilde{\mathfrak{u}}^\sharp$-) module $\nabla(\lambda)$ (resp. $\tilde{\nabla}(\lambda)$) is an injective hull of λ in the category of integrable $\mathfrak{u}^\sharp$- (resp. $\tilde{\mathfrak{u}}^\sharp$-) modules.*

(1.4) Let ζ be the image of v in $\mathcal{K}$, and let U_ζ be the De Concini-Kac algebra [DK] over $\mathcal{K}$ associated to the Cartan matrix A with the generators E_i, F_i, $K_i^{\pm 1}$, $i \in [1,n]$, and the same relations for $\mathbf{U}$ with v replaced by ζ. For each $\alpha \in R^+$ let E_α (resp. F_α) be the root vector of U_ζ associated with α (resp. $-\alpha$), and let $\overline{U}_\zeta = U_\zeta/(E_\alpha^\ell, F_\alpha^\ell \mid \alpha \in R^+)$. If $\overline{U}_\zeta^\flat$ is the $\mathcal{K}$-subalgebra of $\overline{U}_\zeta$ generated by all F_α, $\alpha \in R^+$, and K_i, $i \in [1,n]$, each $\lambda \in \Lambda$ defines a 1-dimensional $\overline{U}_\zeta^\flat$-module by annihilating all F_α and letting K_i act by $\zeta^{d_i\lambda_i}$. Then $\overline{U}_\zeta \otimes_{\overline{U}_\zeta^\flat} \lambda$ comes equipped with a structure of $\tilde{\mathfrak{u}}$-module [AJS, 2.10] such that each $x \in U_\mathcal{K}^0$ acts on $\prod_{\alpha\in R^+} E_\alpha^{c_\alpha} \otimes 1$, $c_\alpha \in \mathbb{N}$, by the scalar $\chi_{\lambda+\sum_\alpha c_\alpha\alpha}(x)$. Put $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

Proposition. *For each $\lambda \in \Lambda$ we have an isomorphism of $\tilde{\mathfrak{u}}$-modules*

$$\tilde{\nabla}(\lambda) \simeq \overline{U}_\zeta \otimes_{\overline{U}_\zeta^\flat} (\lambda - 2(\ell-1)\rho).$$

Proof: Let $\varepsilon_\lambda \in \tilde{\nabla}(\lambda)$ be the element induced by the counit of $\mathfrak{u}^+$. By the universality of $\tilde{\nabla}(\lambda)$ [APW, 0.8.1] there is a homomorphism of $\tilde{\mathfrak{u}}$-modules

(1) $$\overline{U}_\zeta \otimes_{\overline{U}_\zeta^\flat} (\lambda - 2(\ell-1)\rho) \to \tilde{\nabla}(\lambda) \quad \text{such that} \quad E^+ \otimes 1 \mapsto \varepsilon_\lambda,$$

where $E^+ = \prod_{\alpha\in R^+} E_\alpha^{\ell-1}$. On the other hand, by [AJS, 4.9]

(2) the $\tilde{\mathfrak{u}}$-socle of $\overline{U}_\zeta \otimes_{\overline{U}_\zeta^\flat} (\lambda - 2(\ell-1)\rho)$ is simple of highest weight λ.

It follows that the map (1) is injective, and hence bijective by dimension.

(1.5) **Corollary.** *Each $\tilde{\nabla}(\lambda)$ (resp. $\nabla(\lambda)$), $\lambda \in \Lambda$, is the projective cover of $\lambda - 2(\ell-1)\rho$ as integrable $\tilde{\mathfrak{u}}^\sharp$- (resp. $\mathfrak{u}^\sharp$-) module.*

(1.6) Because of the isomorphism (1.4) we call $\tilde{\nabla}(\lambda)$ and also by abuse of language $\nabla(\lambda)$ the infinitesimal Verma module of highest weight λ. By [AJS, 6.3 and 4.10.1]

(1) $\tilde{\nabla}(\lambda)$ (resp. $\nabla(\lambda)$) is simple as $\tilde{\mathfrak{u}}$- (resp. $\mathfrak{u}$-) module iff $\lambda \equiv (\ell-1)\rho \mod \ell\Lambda$.

Let $\mathfrak{u}^{++}$ (resp. $\mathfrak{u}^{--}$) be the augmentation ideal of $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$). If M is a $\mathfrak{u}^\pm$-module, let $M^{\mathfrak{u}^{\pm\pm}}$ denote the annihilater of $\mathfrak{u}^{\pm\pm}$ in M. By (1.3)

(2) $$\tilde{\nabla}(\lambda)^{\mathfrak{u}^{++}} = \nabla(\lambda)^{\mathfrak{u}^{++}} = \lambda.$$

If $\lambda \equiv (\ell-1)\rho \mod \ell\Lambda$, then $\nabla((\ell-1)\rho + \ell\nu) = \nabla((\ell-1)\rho)$, $\nu \in \Lambda$, is simple, called the Steinberg module, and hence

(3) $$\nabla((\ell-1)\rho + \ell\nu)^{\mathfrak{u}^{--}} = -(\ell-1)\rho + \ell\nu.$$

In general, the lowest weight of $\tilde{\nabla}(\lambda)$ (resp. the socle of $\tilde{\nabla}(\lambda)$) is $\lambda - 2(\ell-1)\rho$ (resp. $w_0\lambda^0 + \ell\lambda^1$ if w_0 is an element of the Weyl group of R such that $w_0R^+ = -R^+$ and if one writes $\lambda = \lambda^0 + \ell\lambda^1$ with $\lambda^0 \in \Lambda^{\mathrm{res}}$ and $\lambda^1 \in \Lambda$ [AJS, 4.2.5]). It follows that

(4) $$\dim \nabla(\lambda)^{\mathfrak{u}^{--}} \geq 2 \text{ unless } \nabla(\lambda) \text{ is } \mathfrak{u}\text{-simple}.$$

(1.7) Let $\mathcal{B} = \mathcal{A}/(\phi_\ell)$ and let $\tilde{\mathfrak{u}}_\mathcal{B}$ be the $\mathcal{B}$-subalgebra of $U \otimes_\mathcal{A} \mathcal{B}$ generated by $E_i \otimes 1$, $F_i \otimes 1$, $K_i \otimes 1$, $\begin{bmatrix} K_i; c \\ r \end{bmatrix} \otimes 1$, $i \in [1, n]$, $c \in \mathbb{Z}$, $r \in \mathbb{N}$. Define its $\mathcal{B}$-subalgebras $\tilde{\mathfrak{u}}_\mathcal{B}$, $\mathfrak{u}_\mathcal{B}$, $\tilde{\mathfrak{u}}^\flat_\mathcal{B}$, and $\mathfrak{u}^\flat_\mathcal{B}$ as for $\tilde{\mathfrak{u}}$. An infinitesimal Verma module may be defined over $\mathcal{B}$; $\tilde{\nabla}_\mathcal{B}(\lambda) = \tilde{\mathfrak{u}}^\flat_\mathcal{B}\mathbf{Mod}(\tilde{\mathfrak{u}}_\mathcal{B}, \lambda)$, $\lambda \in \Lambda$, admits a structure of $\tilde{\mathfrak{u}}_\mathcal{B}$-module like $\tilde{\nabla}(\lambda)$, and we have an isomorphism of $\tilde{\mathfrak{u}}$-modules

$$\tilde{\nabla}_\mathcal{B}(\lambda) \otimes_\mathcal{B} \mathcal{K} \simeq \tilde{\nabla}(\lambda).$$

Restricting the $\tilde{\mathfrak{u}}_\mathcal{B}$-action to $\mathfrak{u}_\mathcal{B}$, one obtains $\mathfrak{u}_\mathcal{B}$-module $\nabla_\mathcal{B}(\lambda) = \mathfrak{u}^\flat_\mathcal{B}\mathbf{Mod}(\mathfrak{u}_\mathcal{B}, \lambda)$.

Assume now that ℓ is a prime p. Then $\mathcal{B}/(v-1)$ is a finite filed $\mathbb{F}_p$ of p-elements. Let G be a simply connected simple algebraic group over $\mathbb{F}_p$ associated to the Cartan matrix A with a Borel subgroup B and a maximal torus T of B both split over $\mathbb{F}_p$ such that the roots of B are $-R^+$. Let G_1 (resp. B_1) be the Frobenius kernel of G (resp. B). If $\mathrm{Dist}(G_1)$ (resp. $\mathrm{Dist}(B_1)$) is the algebra of distibutions of G_1 (resp. B_1), there are isomorphisms of $\mathbb{F}_p$-algebras [L2]

$$\mathfrak{u}_\mathcal{B}/(K_i - 1 \mid i \in [1, n]) \otimes_\mathcal{B} \mathbb{F}_p \simeq \mathrm{Dist}(G_1), \quad \mathfrak{u}^\flat_\mathcal{B}/(K_i - 1 \mid i \in [1, n]) \otimes_\mathcal{B} \mathbb{F}_p \simeq \mathrm{Dist}(B_1),$$

and each $\tilde{\nabla}_p(\lambda) := \tilde{\nabla}_\mathcal{B}(\lambda) \otimes_\mathcal{B} \mathbb{F}_p$, $\lambda \in \Lambda$, admits a structure of G_1T-module (cf. [J, II.9]):

$$\tilde{\nabla}_p(\lambda) \simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1)} (\lambda - 2(p-1)\rho).$$

Likewise $\nabla_\mathcal{B}(\lambda) \otimes_\mathcal{B} \mathbb{F}_p$ yields a G_1-module, which we will denote by $\nabla_p(\lambda)$.

2° Maximal cyclic modules

In this section we assume that our Cartan matrix is of type A_n. Then all $d_i = 1$, and we will suppress i from $[\ \ ]_i$. By properly specializing the parametres a maximal cyclic module $\mathcal{V}$ for U_ζ of [DJMM] factors through $\mathfrak{u}$, having a unique, up to scalar, $\mathfrak{u}^+$-primitive vector. Thus $\mathcal{V}$ is of dimension $\ell^{|R^+|}$ and has a simple $\mathfrak{u}$-socle, inviting us to compare $\mathcal{V}$ with infinitesimal Verma modules.

(2.1) Fix $\lambda \in \Lambda$. We define as we may $\mathcal{V}$ to be a $\mathcal{K}$-linear space of basis $u_\mathbf{m}$, $\mathbf{m} \in (\mathbb{Z}/\ell\mathbb{Z})^{|R^+|}$. After [N] we reindex R^+ by the pairs (i, j), $1 \le i \le j \le n$, and we will denote the (i, j)-component of $\mathbf{m}$ by m_{ij}. Then $\mathcal{V}$ admits a structure of integrable $\mathfrak{u}$-module as proved in [N, 5.2] such that for each $i \in [1, n]$ and $\mathbf{m} \in (\mathbb{Z}/\ell\mathbb{Z})^{|R^+|}$

$$(1) \qquad E_i u_\mathbf{m} = \sum_{k=i}^{n} [m_{ik} + m_{i,k-1} - m_{i-1,k-1} - m_{i+1,k}] u_{\mathbf{m}+\epsilon(i,k)+\cdots+\epsilon(i,n)},$$

$$(2) \qquad F_i u_\mathbf{m} = \sum_{k=1}^{i} [-\lambda_i + m_{i+1-k,n-k} - m_{i+1-k,n+1-k} + m_{i-k,n+1-k} - m_{i-k,n-k}] \\ u_{\mathbf{m}-\epsilon(i+1-k,n+1-k)-\epsilon(i+2-k,n+2-k)-\cdots-\epsilon(i,n)},$$

$$(3) \qquad K_i u_\mathbf{m} = \zeta^{\lambda_i + 2m_{in} - m_{i-1,n} - m_{i+1,n}} u_\mathbf{m},$$

where $[r] = \frac{\zeta^r - \zeta^{-r}}{\zeta - \zeta^{-1}}$, $\epsilon(i, j) \in (\mathbb{Z}/\ell\mathbb{Z})^{|R^+|}$ such that $\epsilon(i, j)_{st} = \delta_{is}\delta_{jt}$ for each s and t, and any meaningless terms in the sums should be read as 0. As the structure of $\mathfrak{u}$-module on $\mathcal{V}$ depends on λ, to be precise, we will denote the $\mathfrak{u}$-module $\mathcal{V}$ by $\mathcal{V}(\lambda)$.

A main theorem of [N] is that $\mathcal{V}(\lambda)$ has a unique, up to $\mathcal{K}^\times$, $\mathfrak{u}^+$-primitive element, i.e.,

(4) $$\mathcal{V}(\lambda)^{\mathfrak{u}^{++}} = \mathcal{K}u_{\vec{0}} \quad \text{with} \quad \vec{0} = (0, \dots, 0),$$

and hence by Engel's theorem

(5) $$\mathcal{V}(\lambda) \text{ has a simple } \mathfrak{u}\text{-socle generated by } u_{\vec{0}}.$$

It also follows from (1.3) by dimension that

Proposition. *There is an isomorphism of* $\mathfrak{u}^\sharp$*-modules* $\mathcal{V}(\lambda) \simeq \nabla(\lambda)$.

(2.2) Recall anitiautomorphism τ of $\mathfrak{u}$ such that

$$E_i \mapsto F_i, \quad F_i \mapsto E_i, \quad K_i \mapsto K_i, \quad \forall i \in [1, n].$$

If M is a $\mathfrak{u}$-module, let M^τ be the $\mathcal{K}$-dual space of M with a $\mathfrak{u}$-action given by

$$xf = f(\tau(x)?), \quad f \in M^*, x \in \mathfrak{u}.$$

Then the isomorphism of $\mathfrak{u}^\sharp$-modules $\mathcal{V}(\lambda) \simeq \nabla(\lambda)$ from (2.1) yields an isomorphism of $\mathfrak{u}^-$-modules

$$\begin{aligned} \mathcal{V}(\lambda)^\tau &\simeq \nabla(\lambda)^\tau \\ &\simeq \{\overline{U}_\zeta \otimes_{\overline{U}_\zeta^\flat} (\lambda - 2(\ell-1)\rho)\}^\tau \quad \text{by (1.4)} \\ &\simeq \overline{U}_\zeta \otimes_{\overline{U}_\zeta^\sharp} \lambda \quad \text{by [AJS, 4.10]} \\ &\simeq \mathfrak{u}^-, \end{aligned}$$

where $\overline{U}_\zeta^\sharp$ is the $\mathcal{K}$-subalgebra of $\overline{U}_\zeta$ generated by all E_α, $\alpha \in R^+$, and K_i, $i \in [1, n]$. It follows from [X, 2.5] again that $\mathcal{V}(\lambda)^\tau$ has a unique, up to $\mathcal{K}^\times$, $\mathfrak{u}^-$-primitive vector, and hence

Corollary. $\mathcal{V}(\lambda)$ *has the same simple* $\mathfrak{u}$*-socle and the same simple* $\mathfrak{u}$*-head head as* $\nabla(\lambda)$.

(2.3) We find, moreover, that

Theorem. *The* $\mathfrak{u}$*-module* $\mathcal{V}(\lambda)$ *has a unique, up to* $\mathcal{K}^\times$, $\mathfrak{u}^-$*-primitive vector* $u_{\mathbf{m}}$ *with*

$$m_{ij} \equiv -\sum_{s=1}^{i} \sum_{t=i}^{j} \lambda_{n+s-t} \mod \ell \quad \forall i \leq j,$$

which has $\mathfrak{u}^0$*-weight* $w_0\lambda$.

Proof: The argument is the same as for (2.1.4) from [N, 4.2]; let

$$\sum_{\mathbf{m} \in (\mathbb{Z}/\ell\mathbb{Z})^{|R^+|}} c_{\mathbf{m}} u_{\mathbf{m}} \in \mathcal{V}(\lambda)^{\mathfrak{u}^{--}}, \quad c_{\mathbf{m}} \in \mathcal{K}.$$

As $F_i \sum c_{\mathbf{m}} u_{\mathbf{m}} = 0$ for all F_i, we obtain successively $c_{\mathbf{m}} = 0$ unless

$$(1) \qquad -\lambda_i + m_{i+1-k,n-k} - m_{i+1-k,n+1-k} + m_{i-k,n+1-k} - m_{i-k,n-k} \equiv 0 \mod \ell$$

for each i and $k \in [1, i]$. If $\sum c_{\mathbf{m}} u_{\mathbf{m}} \neq 0$, the system (1) of equations determines $\mathbf{m}$ with $c_{\mathbf{m}} \neq 0$ uniquely as asserted.

(2.4) **Corollary.** *Let* $\lambda \in X$.

(i) *There is an isomorphism of* $\mathfrak{u}^\flat$*-modules*

$$\mathcal{V}(\lambda) \simeq \overline{U}_\zeta \otimes_{\overline{U}_\zeta^\sharp} (w_0\lambda + 2(\ell-1)\rho).$$

(ii) *The* $\mathfrak{u}$*-module* $\mathcal{V}(\lambda)$ *lifts to an integrable* $\tilde{\mathfrak{u}}$*-module iff* $\lambda \equiv (\ell-1)\rho \mod \ell\Lambda$. *In particular, unless* $\lambda \equiv (\ell-1)\rho \mod \ell\Lambda$, $\mathcal{V}(\lambda)$ *is not isomorphic as* $\mathfrak{u}$*-module to any infinitesimal Verma module.*

Proof: For (i) argue as in (2.1) and (2.2).

(ii) If $\lambda \equiv (\ell-1)\rho \mod \ell\Lambda$, the simple $\mathfrak{u}$-socle of $\mathcal{V}(\lambda)$ has dimension $\ell^{|R^+|} = \dim \mathcal{V}(\lambda)$ and hence the assertion follows. We may therefore assume that $\lambda \not\equiv (\ell-1)\rho \mod \ell\Lambda$.

Just suppose $\mathcal{V}(\lambda)$ lift to an integrable $\tilde{\mathfrak{u}}$-module. Then we would have from (1.3) an isomorphism of $\tilde{\mathfrak{u}}^\sharp$-modules

$$\mathcal{V}(\lambda) \simeq \tilde{\nabla}(\lambda + \ell\nu) \quad \text{for some } \nu \in \Lambda.$$

Then the unique $\mathfrak{u}^-$-primitive in $\mathcal{V}(\lambda)$ should have by (1.4) weight $\lambda + \ell\nu - 2(\ell-1)\rho$. That would yield, arguing as in (2.3), an isomorphism of $\tilde{\mathfrak{u}}^\flat$-modules

$$\mathcal{V}(\lambda) \simeq \overline{U}_\zeta \otimes_{\overline{U}_\zeta^\sharp} (\lambda + \ell\nu).$$

Then $\mathcal{V}(\lambda) = \mathfrak{u}^- u_{\vec{0}}$ as $u_{\vec{0}}$ is a highest weight vector of $\mathcal{V}(\lambda)$. But $\mathfrak{u}^- u_{\vec{0}} = \mathfrak{u} u_{\vec{0}}$ is the simple socle of $\mathcal{V}(\lambda)$ and of dimension $< \dim \mathcal{V}(\lambda)$ by (1.6.1), absurd.

(2.5) Assume now that ℓ is a prime p and let $\mathcal{B} = \mathcal{A}/(\phi_p) = \mathbb{Z}[v]/(\phi_p)$ as in (1.7). By construction $\mathcal{V}(\lambda)$ may be defined over $\mathcal{B}$: let $\mathcal{V}_{\mathcal{B}}(\lambda)$ be the free $\mathcal{B}$-module of basis $u_{\mathbf{m}}$, $\mathbf{m} \in (\mathbb{Z}/p\mathbb{Z})^{|R^+|}$, with the $\mathfrak{u}_{\mathcal{B}}$-action given by (2.1.1-3). Regarding $\mathbb{F}_p$ as the quotient $\mathcal{B}/(v-1)$, let $\mathcal{V}_p(\lambda) = \mathcal{V}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{B}} \mathbb{F}_p$. Then $\mathcal{V}_p(\lambda)$ is naturally a G_1-module for $G = \mathrm{SL}_{n+1} \otimes_{\mathbb{Z}} \mathbb{F}_p$ in the setup of (1.7). The proof of (2.1.4) from [N] and the argument of (2.3) carry over to obtain

Theorem. *Let* $\lambda \in \Lambda$

(i) *There is an isomorphism of* B_1^+*-modules* $\mathcal{V}_p(\lambda) \simeq \nabla_p(\lambda)$.

(ii) $\mathcal{V}_p(\lambda)$ *has a unique, up to* $\mathbb{F}_p^\times$, B_1^+*- and* B_1*-invariant vector, respectively, and hence has a simple* G_1*-socle and a simple* G_1*-head, where* B_1^+ *is the Frobenius kernel of the Borel subgroup* B^+ *of* G *opposite to* B.

(iii) *If* $\lambda \not\equiv (p-1)\rho \mod p\Lambda$*, the structure of* G_1*-module on* $\mathcal{V}_p(\lambda)$ *does not lift to* G_1T*-module, and hence* $\mathcal{V}_p(\lambda)$ *is not isomorphic to any infinitesimal Verma module as* G_1*-module.*

(2.6) **Remark.** If the simple $\mathfrak{u}$-module generated by $u_{\vec{0}}$ in $\mathcal{V}(\lambda)$ and the simple G_1-module generated by $u_{\vec{0}} \otimes 1$ in $\mathcal{V}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{B}} \mathbb{F}_p$ have the same dimension, then Lusztig's conjecture for the irreducible $\mathrm{SL}_{n+1} \otimes_{\mathbb{Z}} \mathbb{F}_p$-modules will follow from [KL] and [KT], [C]; for $p \geq n+1$ the converse is also expected to hold.

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