h-VECTORS OF GORENSTEIN* SIMPLICIAL POSETS

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1. Introduction

A *simplicial poset* P (also called a *boolean poset* and a *poset of boolean type*) is a finite poset with a smallest element $\hat{0}$ such that every interval $[\hat{0}, y]$ for $y \in P$ is a boolean algebra, i.e., $[\hat{0}, y]$ is isomorphic to the set of all subsets of a finite set, ordered by inclusion. The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the *face poset* of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex. Although a simplicial poset P is not necessarily the face poset of a simplicial complex, it is always the face poset of a CW-complex $\Gamma(P)$. In fact, to each $y \in P\backslash\{\hat{0}\} = \overline{P}$, we assign a (geometrical) simplex whose face poset is $[\hat{0}, y]$ and glue those geometrical simplices according to the order relation in P. Then we get the CW-complex $\Gamma(P)$ such that all the attaching maps are inclusions. For instance, if two simplicies of a same dimension are identified on their boundaries via the identity map, then it is not a simplicial complex but a CW-complex obtained from a simplicial poset. The CW-complex $\Gamma(P)$ has a well-defined barycentric subdivision which is isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P}$. Here $\Delta(\overline{P})$ is a simplicial complex on the vertex set $\overline{P}$ whose faces are the chains of $\overline{P}$.

We say that $y \in P$ has rank i if the interval $[\hat{0}, y]$ is isomorphic to the boolean algebra of rank i (in other words, the face poset of an $(i-1)$-simplex), and the rank of P is defined to be the maximum of ranks of all elements in P. Let $d = \operatorname{rank} P$. In exact analogy to simplicial complexes, the f-vector of the simplicial poset P, $(f_0, f_1, \dots, f_{d-1})$, is defined by

$$f_i = f_i(P) = \sharp\{y \in P \mid \operatorname{rank} y = i + 1\}$$

and the h-vector of P, $(h_0, h_1, \dots, h_d)$, is defined by the following identity:

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}, \tag{1.1}$$

where $f_{-1} = 1$, so $h_0 = 1$. When P is the face poset of a simplicial complex Σ, the f- and h-vector of P coincide with the classical f- and h-vector of the simplicial complex Σ respectively. f-vectors and h-vectors have equivalent information, but h-vectors are often easier than f-vectors. In [8], R. Stanley discussed characterization of h-vectors for certain classes of simplicial posets. For example, he proved that a vector $(h_0, h_1, \ldots, h_d)$ of integers with $h_0 = 1$ is the h-vector of a Cohen-Macaulay simplicial poset of rank d if and only if $h_i \geq 0$ for all i. As for Gorenstein* simplicial posets of rank d, it is known that h-vectors must satisfy Dehn-Sommerville equations $h_i = h_{d-i}$ for all i, in addition to the non-negativity conditions $h_i \geq 0$. In this paper we will prove that h-vectors of Gorenstein* simplicial posets must satisfy one more subtle condition conjectured by Stanley in [8], see [1], [5], [8] for partial results.

Theorem 1.1. *Suppose that $(h_0, h_1, \ldots, h_d)$ is the h-vector of a Gorenstein* simplicial poset of rank d. Then $h_{d/2}$ must be even if d is even and $h_i = 0$ for some $i > 0$.*

Combining this with Theorem 4.3 in [8], one completes characterization of h-vectors of Gorenstein* simplicial posets.

Corollary 1.2. *Let $(h_0, h_1, \ldots, h_d)$ be a vector of non-negative integers with $h_i = h_{d-i}$ for all i and $h_0 = 1$. Any of the following (mutually exclusive) conditions are necessary and sufficient for the existence of a Gorenstein* simplicial poset P of rank d with $h_i(P) = h_i$ for all i:*

(1) *d is odd,*
(2) *d is even and $h_{d/2}$ is even,*
(3) *d is even, $h_{d/2}$ is odd, and $h_i > 0$ for all i.*

Our proof of Theorem 1.1 is purely algebraic but the idea stems from topology, so we will explain how our proof is related to topology in Section 2. A main tool to study the h-vector of a simplicial poset P is a (generalized) face ring A_P introduced in [8] of the poset P. In Section 3 we discuss restriction maps from A_P to polynomial rings. In Section 4 we construct a map called an index map from A_P to a polynomial ring. Theorem 1.1 is proven in Section 5.

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2. Relation to topology

Like simple convex polytopes are closely related to objects (in algebraic geometry) called toric manifolds or orbifolds (see [2]), Gorenstein* simplicial posets are closely related to objects (in topology) called torus manifolds or orbifolds (see [4], [5]), and the proof of Theorem 1.1 is motivated by a topological observation described in this section. A special class of Gorenstein* simplcial posets is treated in [5] with a similar idea.

We shall illustrate relations between combinatorics and topology with simple examples. In the following, T will denote the product of d copies of the circle group consisting of complex numbers with unit length, i.e., T is a d-dimensional torus group.

Example 2.1. A complex projective space $\mathbb{C}P^d$ has a T-action defined in the homogeneous coordinates by

$$(t_1,\dots,t_d)\cdot(z_0:z_1:\dots:z_d)=(z_0:t_1z_1:\dots:t_dz_d).$$

The orbit space $\mathbb{C}P^d/T$ has a natural face structure. In fact, facets are images of codimension two submanifolds $z_i=0$ $(i=0,1,\dots,d)$ by the quotient map $\mathbb{C}P^d\to\mathbb{C}P^d/T$. The map (called a moment map)

$$(z_0:z_1:\dots:z_d)\to\frac{1}{\sum_{i=0}^d|z_i|^2}(|z_1|^2,\dots,|z_d|^2)$$

induces a face preserving homeomorphism from the orbit space $\mathbb{C}P^d/T$ to a standard d-simplex, which is a simple convex polytope. The face poset of $\mathbb{C}P^d/T$ with respect to reversed inclusion (so $\mathbb{C}P^d/T$ iteself is the smallest element) is the face poset of a simplicial complex of dimension $d-1$ and Gorenstein*.

Similarly, the product of d copies of $\mathbb{C}P^1$ admits a T-action, the orbit space $(\mathbb{C}P^1)^d/T$ is homeomorphic to a d-cube, which is also a simple convex polytope, and the face poset of $(\mathbb{C}P^1)^d/T$ is also the face poset of a simplcial complex of dimension $d-1$ and Gorenstein*.

Example 2.2. Let S^{2d} be the $2d$-sphere identified with the following subset in $\mathbb{C}^d\times\mathbb{R}$:

$$\{(z_1,\dots,z_d,y)\in\mathbb{C}^d\times\mathbb{R}\mid|z_1|^2+\dots+|z_d|^2+y^2=1\},$$

and define a T-action on S^{2d} by

$$(t_1,\dots,t_d)\cdot(z_1,\dots,z_d,y)=(t_1z_1,\dots,t_dz_d,y).$$

Then facets in the orbit space S^{2d}/T are images of codimension two submanifolds $z_i=0$ $(i=1,\dots,d)$ by the quotient map $S^{2d}\to S^{2d}/T$, and the map

$$(z_1,\dots,z_d,y)\to(|z_1|,\dots,|z_d|,y)$$

induces a face preserving homeomorphism from S^{2d}/T to this subset of the d-sphere:

$$\{(x_1,\dots,x_d,y)\in\mathbb{R}^{d+1}\mid x_1^2+\dots+x_d^2+y^2=1,\ x_1\ge0,\dots,x_d\ge0\}.$$

The orbit space S^{2d}/T is not (isomorphic to) a simple convex polytope because the intersection of d facets consists of two points, but it is a manifold with corners and every face (even S^{2d}/T itself) is acyclic. The face poset of S^{2d}/T is not the face poset of any simplicial complex. However, it is a simplicial poset and Gorenstein*. The geometric realization of the face poset of S^{2d}/T is formed from two $(d-1)$-simplices by gluing their boundaries via the identity map.

A projective toric orbifold is related to a simple convex polytope through the quotient (or moment) map as in Example 2.1, and the h-vector of the simple convex polytope agrees with the (even degree) betti numbers of the toric orbifold. Noting this fact, Stanley [6] deduced constraint on the h-vector by applying the hard Lefschetz theorem to the toric orbifold and completed characterization of h-vectors of simple (or simplicial) convex polytopes.

In some sense our proof of Theorem 1.1 is on this line. The argument discussed below in this section is not completely verified but would be helpful for the reader to understand what is done in subsequent sections. A projective toric orbifold is associated with a simple convex polytope and conversely a projective toric orbifold determines a simple convex polytope through the quotient (or moment) map. This correspondence exists in a more extended category. We note that a simple convex polytope determines a Gorenstein* simplicial complex (as its dual complex) together with a linear system of parameters (abbreviated as an l.s.o.p.) of the face ring (over $\mathbb{Q}$) of the simplicial complex. As is discussed in [5], a torus orbifold M (introduced in [4]) with vanishing odd degree cohomology over $\mathbb{Q}$ would be associated with a Gorenstein* simplicial poset P together with an l.s.o.p. of the face ring A_P (over $\mathbb{Q}$) of P, and conversely a torus orbifold with vansihing odd degree cohomology over $\mathbb{Q}$ would determine a Gorenstein* simplicial poset through the quotient map. (In fact, this is established in [5] for Gorenstein* simplicial posets with l.s.o.p. over $\mathbb{Z}$, and in this case the associated torus orbifold is smooth, so it is a torus manifold. But, in order to treat Gorenstein* simplicial posets over an arbitrary filed, we need to develope the argument over $\mathbb{Q}$, so orbifolds will appear.) When P with an l.s.o.p. of A_P comes from a simple convex polytope, we may take M to be a toric orbifold. The torus orbifold M is an orbifold of dimension $2d$ with a T-action and would have these properties:

Properties.

(1) $H^{odd}(M;\mathbb{Q}) = 0$,
(2) $h_i(P)$ agrees with the $2i$-th betti number $b_{2i}(M)$ of M,
(3) the equivariant cohomology ring $H^*_T(M;\mathbb{Q})$ of M is isomorphic to A_P.

(These properties are established for torus manifolds in [5] with $\mathbb{Z}$ instead of $\mathbb{Q}$.) Here $H^*_T(M;\mathbb{Q})$ is defined as

$$H^*_T(M;\mathbb{Q}) := H^*(ET \times_T M;\mathbb{Q})$$

where ET is the total space of the universal principal T-bundle (on which T acts freely) and $ET \times_T M$ is the orbit space of the product $ET \times M$ by the diagonal T-action. In short, the above discussion tells us that characterization of h-vectors of Gorenstein* simplicial posets would be equivalent to that of (even degree) betti numbers of torus orbifolds with property (1).

The sufficiency of Corollary 1.2 is proved in [8] and it can be observed from our point of view as follows. Since products of torus orbifolds are also torus orbifolds, $S^{2d-2k} \times S^{2k}$ $(1 \le k \le d-1)$ are torus orbifolds, in fact, they are torus manifolds because they are smooth. They satisfy property (1). If M_1 and M_2 are torus manifolds of a same dimension with property (1), then their equivariant connected sum $M_1 \sharp M_2$ at fixed points (having isomorphic tangential representations) produces a torus manifold with property (1) and

$$b_{2i}(M_1 \sharp M_2) = b_{2i}(M_1) + b_{2i}(M_2) \qquad \text{for } 1 \le i \le d-1.$$

Therefore, if we take S^{2d} or equivariant connected sum of a finite number of $\mathbb{C}P^d$ and $S^{2d-2k} \times S^{2k}$, then we see that any vector satisfying the conditions in Corollary 1.2 can be realized as a vector of (even degree) betti numbers of a torus manifold with property (1). This proves the sufficiency of Corollary 1.2 because a torus manifold with property (1) determines a Gorenstein* simplicial poset through the orbit space and they satisfy property (2).

As is shown above, one can use topological techniques or ideas to study h-vectors of Gorenstein* simplicial posets. What we will use to deduce the necessity in Theorem 1.1 is the *index map* in equivariant cohomology:

$$\mathrm{Ind}_T \colon H^*_T(M;\mathbb{Q}) \to H^{*-2d}_T(pt;\mathbb{Q}) = H^{*-2d}(BT;\mathbb{Q}),$$

where $BT = ET/T$ is the classifying space of principal T-bundles. The index map is nothing but the Gysin homomorphism in equivariant cohomology induced from the collapsing map $\pi\colon M \to pt$. As is well-known, BT is the product of d copies of $\mathbb{C}P^\infty$ (up to homotopy) and $H^*(BT;\mathbb{Q})$ is a polynomial ring in d variables of degree two. The index map Ind_T decreases cohomological degrees by $2d$ because the dimension of M is $2d$. Moreover, $H^*_T(M;\mathbb{Q})$ is a module over $H^*(BT;\mathbb{Q})$ through $\pi^*\colon H^*(BT;\mathbb{Q}) = H^*_T(pt;\mathbb{Q}) \to H^*_T(M;\mathbb{Q})$ and Ind_T is an $H^*(BT;\mathbb{Q})$-module map. Since $H^{odd}(M;\mathbb{Q}) = 0$ and $H^*(BT;\mathbb{Q})$ is a polynomial ring in d variables, say $t_1,\dots,t_d$, the quotient ring of $H^*_T(M;\mathbb{Q})$ by the ideal generated by $\pi^*(t_1),\dots,\pi^*(t_d)$ agrees with the ordinary cohomology $H^*(M;\mathbb{Q})$. Similarly, the quotient ring of $H^*_T(pt;\mathbb{Q}) = H^*(BT;\mathbb{Q})$ by the ideal generated by $t_1,\dots,t_d$ agrees with $H^*(pt;\mathbb{Q})$. Therefore the index map in equivariant cohomology induces the index map in ordinary cohomology:

$$\mathrm{Ind}\colon H^*(M;\mathbb{Q}) \to H^{*-2d}(pt;\mathbb{Q}).$$

This map agrees with the Gysin homomorphism in ordinary cohomology induced from the collapsing map π, so it is the evaluation map on a fundamental class of M. Thus, we have a commutative diagram:

$$\begin{array}{ccc}
H^{2d}_T(M;\mathbb{Q}) & \xrightarrow{\mathrm{Ind}_T} & H^0_T(pt;\mathbb{Q}) = H^0(BT;\mathbb{Q}) = \mathbb{Q} \\
\downarrow & & \downarrow \\
H^{2d}(M;\mathbb{Q}) & \xrightarrow{\mathrm{Ind}} & H^0(pt;\mathbb{Q}) = \mathbb{Q},
\end{array}$$

where the right vertical map is the identity. A key thing is to find an element ω_T in $H^{2d}_T(M;\mathbb{Q})$ such that

(i) ω_T is a polynomial in elements of $H^2_T(M;\mathbb{Q})$,

(ii) $\mathrm{Ind}_T(\omega_T)$ is an integer and $\mathrm{Ind}_T(\omega_T) \equiv \chi(M) \pmod 2$, where $\chi(M)$ is the Euler characteristic of M.

We may think of ω_T as a "lifting" of the equivariant top Stiefel-Whitney class $w^T_{2d}(M) \in H^{2d}_T(M;\mathbb{Z}/2)$ of M. If we find such an element ω_T, then it follows from the commutativity of the above diagram that

$$\mathrm{Ind}_T(\omega_T) = \mathrm{Ind}(\omega) \tag{2.1}$$

where ω is the image of ω_T by the left vertical map in the above diagram.

Now suppose $h_i(P) = 0$ for some $i > 0$. Then the $2i$-th betti number $b_{2i}(M)$ of M is zero by property (2) and the element ω vanishes because it is a polynomial in degree two elements by (i) above, so the right hand side at (2.1) is zero and $\chi(M)$ is even by (ii) above. On the other hand, it follows from properties (1) and (2) that

$$\chi(M) = \sum_{i=0}^{d} b_{2i}(M) = \sum_{i=0}^{d} h_i(P).$$

Since Dehn-Sommerville equations $h_i(P) = h_{d-i}(P)$ hold for all i (which follow from the Poincaré duality for M and property (2)), the fact that $\chi(M)$ is even means that $h_{d/2}(P)$ is even when d is even.

It turns out that the argument developed above works without assuming the existence of the torus orbifold M. In fact, the face ring A_P takes the place of $H^*_T(M;\mathbb{Q})$ by property (3) and an l.s.o.p. for A_P plays the role of $\pi^*(t_1), \dots, \pi^*(t_d)$ so that the polynomial ring generated by the l.s.o.p. corresponds to the polynomial ring $\pi^*(H^*(BT;\mathbb{Q}))$ (or $H^*(BT;\mathbb{Q})$ since π^* is injective). The index map Ind_T has an expression (so-called Lefschetz fixed point formula) in terms of local data around T-fixed points of M, and since the formula is purely algebraic, one can use it to define an "index map" from A_P. To carry out this idea, we need to study restriction maps from A_P to polynomial rings because restriction maps to T-fixed points in equivariant cohomology are involved in the Lefschetz fixed point formula. We will discuss such restriction maps in Section 3 and construct the index map from A_P in Section 4.

3. Restriction maps

In this and next sections, we consider rings over $\mathbb{Q}$. A main tool to study the h-vector of a (finite) simplicial poset P is the face ring A_P of the poset P introduced by Stanley in [8]. We recall it first.

Definition. Let P be a simplcial poset of rank d with elements $\hat{0}, y_1, \dots, y_p$. Let $A = \mathbb{Q}[y_1, \dots, y_p]$ be the polynomial ring over $\mathbb{Q}$ in the variables y_i and define $\mathcal{I}_P$ to be the ideal of A generated by the following elements:

$$y_i y_j - (y_i \wedge y_j)(\sum_z z),$$

where $y_i \wedge y_j$ is the greatest lower bound of y_i and y_j, z ranges over all minimal upper bounds of y_i and y_j, and we understand $\sum_z z = 0$ if y_i and y_j have no common upper bound. Then the face ring A_P of the simplicial poset P is defined as the quotient ring $A/\mathcal{I}_P$ and made graded

$$A_P = (A_P)_0 \oplus (A_P)_1 \oplus \cdots \oplus (A_P)_d$$

by defining $\deg y_i = \operatorname{rank} y_i$. The ring A_P reduces to a classical Stanley-Reisner face ring when P is the face poset of a simplicial complex.

We denote by P_s the subset of P consisting of elements of rank s. Elements in P_1 will be denoted by $x_1, \dots, x_n$ and called *atoms* in P, so $x_1, \dots, x_n$ is a basis of $(A_P)_1$.

Suppose that y is an element of P_d. Then the interval $[\hat{0}, y]$ is a boolean algebra of rank d and $A_{[\hat{0},y]}$ is a polynomial ring in d variables. Sending all elements in P which are not lower than y to zero, we obtain an epimorphism

$$\iota_y \colon A_P \to A_{[\hat{0},y]}.$$

Since $\mathbb{Q}$ is a field with infinitely many elements, A_P admits an l.s.o.p. $\theta_1, \dots, \theta_d$ (see the proof of Theorem 3.10 in [8]). In the following we fix the l.s.o.p.

Lemma 3.1. *The restriction of ι_y to the polynomial subring $\mathbb{Q}[\theta_1, \dots, \theta_d]$ of A_P:*

$$\iota_y \colon \mathbb{Q}[\theta_1, \dots, \theta_d] \to A_{[\hat{0},y]}$$

is an isomorphism.

Proof. Since A_P is finitely generated as a $\mathbb{Q}[\theta_1, \dots, \theta_d]$-module, so is $A_{[\hat{0},y]}$. This implies that ι_y maps the vector space spanned by $\theta_1, \dots, \theta_d$ isomorphically onto the vector space spanned by d elements of degree one generating the polynomial ring $A_{[\hat{0},y]}$, thus the lemma follows. □

Henceforth, we identify $A_{[\hat{0},y]}$ with $\mathbb{Q}[\theta_1, \dots, \theta_d]$ via ι_y, and think of ι_y as a map from A_P to $\mathbb{Q}[\theta_1, \dots, \theta_d]$. Note that ι_y is a $\mathbb{Q}[\theta_1, \dots, \theta_d]$-module map.

We investigate $\iota_y(x_j)$ for atoms x_j. Denote by Θ the vector space of dimension d spanned by $\theta_1, \dots, \theta_d$ over $\mathbb{Q}$, and by Θ^* its dual space. Note that Θ is a vector subspace of $(A_P)_1$.

Lemma 3.2. *There is a unique element $\gamma_i \in \Theta^*$ for each i such that*

$$\theta = \sum_{i=1}^{n} \langle \gamma_i, \theta \rangle x_i \quad \text{for any } \theta \in \Theta,$$

where $\langle \ , \ \rangle$ is a natural pairing between Θ^* and Θ.

Proof. Since the atoms $x_1, \dots, x_n$ form a basis of $(A_P)_1$ over $\mathbb{Q}$, there is a unique rational number $r_i(\theta)$ for each i (depending on θ) such that

$$\theta = \sum_{i=1}^{n} r_i(\theta) x_i.$$

Clearly $r_i(\theta)$ is linear with respect to θ, so there is a unique $\gamma_i \in \Theta^*$ for each i such that $r_i(\theta) = \langle \gamma_i, \theta \rangle$. □

For $w \in P_s$, we set

$$\mathcal{A}(w) := \{i \in \{1, \dots, n\} \mid x_i \text{ is an atom lower than } y\}.$$

The cardinality of $\mathcal{A}(w)$ is s.

Lemma 3.3. *Let $y \in P_d$.*

(1) *$\{\gamma_i \in \Theta^* \mid i \in \mathcal{A}(y)\}$ is a basis of Θ^*.*
(2) *If $\{\theta_i(y) \in \Theta \mid i \in \mathcal{A}(y)\}$ is the dual basis of $\{\gamma_i \in \Theta^* \mid i \in \mathcal{A}(y)\}$, then*

$$\iota_y(x_i) = \begin{cases} \theta_i(y) & i \in \mathcal{A}(y), \\ 0 & i \notin \mathcal{A}(y). \end{cases}$$

Proof. (1) Sending the identity in Lemma 3.2 by ι_y, we get

$$\iota_y(\theta) = \sum_{i \in \mathcal{A}(y)} \langle \gamma_i, \theta \rangle \iota_y(x_i) \tag{3.1}$$

because $\iota_y(x_i) = 0$ for $i \notin \mathcal{A}(y)$ by the definition of ι_y. If $\{\gamma_i \mid i \in \mathcal{A}(y)\}$ is not a basis of Θ^*, then there exists a non-zero θ such that $\langle \gamma_i, \theta \rangle = 0$ for all $i \in \mathcal{A}(y)$, so that $\iota_y(\theta) = 0$. But this contradicts Lemma 3.1.

(2) If $i \notin \mathcal{A}(y)$, then $\iota_y(x_i) = 0$ as remarked above. If $i \in \mathcal{A}(y)$, then we take $\theta = \theta_i(y)$ in the identity (3.1), so that $\iota_y(\theta_i(y)) = \iota_y(x_i)$. Since $\mathbb{Q}[\theta_1, \dots, \theta_d]$ is identified with $A_{[\hat{0},y]}$ via ι_y, $\iota_y(\theta_i(y))$ is identified with $\theta_i(y)$. Therefore the lemma is proven. □

For $z \in P_{d-1}$, let y, y' be elements in P_d upper than z. We define $\ell, \ell' \in \{1, \dots, n\}$ by

$$\mathcal{A}(y) \backslash \mathcal{A}(z) = \{\ell\}, \qquad \mathcal{A}(y') \backslash \mathcal{A}(z) = \{\ell'\}.$$

It may happen that $\ell = \ell'$. Since $\{\gamma_i \mid i \in \mathcal{A}(y)\}$ and $\{\gamma_i \mid i \in \mathcal{A}(y')\}$ are both bases of Θ^*, one has an expression

$$\gamma_{\ell'} = b\gamma_\ell + \sum_{i \in \mathcal{A}(z)} b_i \gamma_i \tag{3.2}$$

with $b \neq 0, b_i \in \mathbb{Q}$.

Lemma 3.4. *The following hold:*

$$\begin{aligned} \theta_i(y') &= \theta_i(y) - \frac{b_i}{b} \theta_\ell(y) \quad \textit{for } i \in \mathcal{A}(z), \\ \theta_{\ell'}(y') &= \frac{1}{b} \theta_\ell(y). \end{aligned}$$

Proof. It suffices to check that $\{\theta_i(y)-\frac{b_i}{b}\theta_\ell(y)\mid i\in\mathcal{A}(z)\}\cup\{\frac{1}{b}\theta_\ell(y)\}$ is the dual basis of $\{\gamma_j\mid j\in\mathcal{A}(y')(=\mathcal{A}(z)\cup\{\ell'\})\}$. When $j\in\mathcal{A}(z)(=\mathcal{A}(y)\backslash\{\ell\})$, we have

$$\begin{aligned}\langle\gamma_j,\theta_i(y)-\frac{b_i}{b}\theta_\ell(y)\rangle&=\delta_{ij},\\ \langle\gamma_j,\frac{1}{b}\theta_\ell(y)\rangle&=0.\end{aligned}$$

When $j=\ell'$, it follows from the identity (3.2) that

$$\begin{aligned}\langle\gamma_{\ell'},\theta_i(y)-\frac{b_i}{b}\theta_\ell(y)\rangle&=\langle b\gamma_\ell+\sum_{i\in\mathcal{A}(z)}b_i\gamma_i,\theta_i(y)-\frac{b_i}{b}\theta_\ell(y)\rangle=0,\\ \langle\gamma_{\ell'},\frac{1}{b}\theta_\ell(y)\rangle&=\langle b\gamma_\ell+\sum_{i\in\mathcal{A}(z)}b_i\gamma_i,\frac{1}{b}\theta_\ell(y)\rangle=1.\end{aligned}$$

This proves the lemma. □

It follows from Lemmas 3.3(2) and 3.4 that

$$\iota_y(x_i)\equiv\iota_{y'}(x_i)\mod\theta_\ell(y)\tag{3.3}$$

for any $i\in\mathcal{A}(y)\cup\mathcal{A}(y')$.

Lemma 3.5. *We have* $\iota_y(\alpha)\equiv\iota_{y'}(\alpha)\mod\theta_\ell(y)$ *for any* $\alpha\in A_P$.

Proof. Since $\iota_y\colon A_P\to A_{[\hat{0},y]}$ and $\iota_{y'}\colon A_P\to A_{[\hat{0},y']}$ factor through $A_{[\hat{0},y]\cup[\hat{0},y']}$, we may assume $P=[\hat{0},y]\cup[\hat{0},y']$. Also, since A_P is generated by elements in P, it suffices to prove the lemma for $\alpha\in P$. Note that

$$\iota_y(\alpha)=\iota_y(\prod_{i\in\mathcal{A}(\alpha)}x_i)\quad\text{if }\alpha\in[\hat{0},y].\tag{3.4}$$

If $\alpha\in[\hat{0},y]\cap[\hat{0},y']$, then the lemma follows from (3.3) and (3.4). Therefore, we may assume that $\alpha\in[\hat{0},y]$ but $\alpha\notin[\hat{0},y']$. In this case $\mathcal{A}(\alpha)$ contains ℓ (otherwise $\alpha\in[\hat{0},z]\subset[\hat{0},y']$), so $\iota_y(\alpha)\equiv0\mod\theta_\ell(y)$ by (3.4) and Lemma 3.3(2). Moreover, $\iota_{y'}(\alpha)=0$ by definition. Therefore the lemma is proven. □

About the absolute value of b in the identity (3.2). Let $\{\theta_i^*\mid1\le i\le d\}$ be the dual basis of $\theta_1,\dots,\theta_d$ and define $m(y)$ to be the absolute value of the determinant of a matrix sending the basis $\{\theta_i^*\mid1\le i\le d\}$ to the basis $\{\gamma_i\mid i\in\mathcal{A}(y)\}$. Then

$$m(y')=|b|m(y).\tag{3.5}$$

by (3.2). Note that

$$m(y)=m(y')\text{ if }\mathcal{A}(y)=\mathcal{A}(y').\tag{3.6}$$

About the sign of b. Give an orientation on Θ^* determined by an ordered basis $(\theta_1^*,\dots,\theta_d^*)$ and choose an order of the basis $\{\gamma_i\mid i\in\mathcal{A}(y)\}$ whose induced orientation on Θ^* agrees with the given orientation. This determines an order of atoms x_i ($i\in\mathcal{A}(y)$) and then determines an orientation on the $(d-1)$-simplex with those atoms as vertices. The oriented $(d-1)$-simplex obtained in this way is denoted by $\langle y\rangle$. Then the boundaries $\partial\langle y\rangle$ and $\partial\langle y'\rangle$ of $\langle y\rangle$ and $\langle y'\rangle$ have opposite orientations on the $(d-2)$-simplex $[z]$ corresponding to z (in other words, $[z]$ does not appear in $\partial\langle y\rangle + \partial\langle y'\rangle$) if and only if $b < 0$.

4. Index maps

In this section, we define an "index map" from A_P to the polynomial ring $\mathbb{Q}[\theta_1,\dots,\theta_d]$, which corresponds to the index map Ind_T in Section 2. It is a $\mathbb{Q}[\theta_1,\dots,\theta_d]$-module map, so it induces a homomorphism from the quotient ring $A_P/(\theta_1,\dots,\theta_d)$ to $\mathbb{Q}$. This induced map corresponds to the index map in ordinary cohomology.

We pose the following assumption.

Assumption.

(1) For any $z \in P_{d-1}$, there are exactly two elements in P_d upper than z.
(2) One can assign a sign $\epsilon(y) \in \{\pm 1\}$ to each $y \in P_d$ so that $\sum_{y\in P_d}\epsilon(y)\langle y\rangle$ is a cycle (hence defines a fundamental class in $H_{d-1}(|\Gamma(P)|;\mathbb{Z})$ where $|\Gamma(P)|$ denotes the underlying space of the CW-complex $\Gamma(P)$ explained in the Introduction).

When $\langle y\rangle$ and $\langle y'\rangle$ share a $(d-2)$-simplex $[z]$, it follows from the above assumption that $[z]$ does not appear in $\partial(\epsilon(y)\langle y\rangle) + \partial(\epsilon(y')\langle y'\rangle)$. Therefore

$$\epsilon(y)\,\epsilon(y') \text{ and } b \text{ have opposite signs} \tag{4.1}$$

by the remark mentioned at the end of Section 3.

We define the *index map* by

$$\mathrm{Ind}_T(\alpha) := \sum_{y\in P_d} \frac{\epsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} \qquad \text{for } \alpha \in A_P. \tag{4.2}$$

Apparently, $\mathrm{Ind}_T(\alpha)$ is a rational function in $\theta_1,\dots,\theta_d$. But, we have

Lemma 4.1. $\mathrm{Ind}_T(\alpha) \in \mathbb{Q}[\theta_1,\dots,\theta_d]$ *for any* $\alpha \in A_P$.

Remark. The proof given below is essentially same as that of Theorem 2.2 in [3]. A similar result can be found in [4, Section 8].

Proof. The right hand side at (4.2) can be expressed as

$$\frac{g}{\prod_{j=1}^{N} f_j} \tag{4.3}$$

with $g \in \mathbb{Q}[\theta_1,\dots,\theta_d]$ and $f_j \in \Theta \subset \mathbb{Q}[\theta_1,\dots,\theta_d]$ such that any two of $f_1,\dots,f_N$ are linearly independent. It suffices to show that f_1 divides g.

Let Q be the set of $y \in P_d$ such that $\theta_i(y)$ is not a scalar multiple of f_1 for every $i \in \mathcal{A}(y)$, and let Q^c be the complement of Q in P_d. In (4.2), the sum of terms for elements in Q reduces to

$$\sum_{y\in Q} \frac{\epsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} = \frac{g_1}{\prod_{j=2}^N f_j} \tag{4.4}$$

with $g_1 \in \mathbb{Q}[\theta_1, \dots, \theta_d]$, so that f_1 does not appear in the denominator.

On the other hand, if $y \in Q^c$, then it follows from the definition of Q that there is an element $\ell \in \mathcal{A}(y)$ such that

$$\theta_\ell(y) = cf_1 \qquad (0 \neq c \in \mathbb{Q}),$$

and there is a unique element $z \in P_{d-1}$ such that z is lower than y and $\mathcal{A}(z) = \mathcal{A}(y)\backslash\{\ell\}$. By assumption, there is a unique element in P_d which is upper than z and different from y. We denote it by y'. Now we are in the same situation as in Lemma 3.4. By Lemma 3.4

$$\theta_{\ell'}(y') = \frac{1}{b}\theta_\ell(y)\big(= (c/b)f_1\big),$$

so y' is also an element in Q^c. Noting that $\mathcal{A}(y) = \mathcal{A}(z) \cup \{\ell\}$ and $\mathcal{A}(y') = \mathcal{A}(z) \cup \{\ell'\}$, we combine the two terms in (4.2) for y and y' to get

$$\begin{aligned} &\frac{\epsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} + \frac{\epsilon(y')\iota_{y'}(\alpha)}{m(y')\prod_{i\in\mathcal{A}(y')}\theta_i(y')} \\ =&\frac{m(y')\,\epsilon(y)\iota_y(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y') + bm(y)\,\epsilon(y')\iota_{y'}(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y)}{m(y)m(y')\theta_\ell(y)\prod_{i\in\mathcal{A}(z)}\theta_i(y)\prod_{i\in\mathcal{A}(z)}\theta_i(y')}. \end{aligned} \tag{4.5}$$

Here

$$\iota_y(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y') \equiv \iota_{y'}(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y) \mod \theta_\ell(y)$$

by Lemmas 3.4 and 3.5, and

$$m(y')\,\epsilon(y) = -bm(y)\,\epsilon(y')$$

by (3.5) and (4.1), so the numerator of the right hand side at the identity (4.5) is divisible by $\theta_\ell(y) = cf_1$. This means that we can arrange the left hand side at the identity (4.5) with a common denominator in which f_1 does not appear as a factor. Since elements in Q^c appear pairwise like this, one has

$$\sum_{y\in Q^c} \frac{\epsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in A(y)}\theta_i(y)} = \frac{g_2}{\prod_{j=2}^N f_j}$$

with $g_2 \in \mathbb{Q}[\theta_1, \dots, \theta_d]$. This together with (4.4) implies that the numerator g in (4.3) is divisible by f_1. □

Since ι_y is a $\mathbb{Q}[\theta_1,\dots,\theta_d]$-module map, so is Ind_T. Therefore

$$\mathrm{Ind}_T\colon A_P \to \mathbb{Q}[\theta_1,\dots,\theta_d]$$

induces a homomorphism

$$\mathrm{Ind}\colon A_P/(\theta_1,\dots,\theta_d) \to \mathbb{Q}. \tag{4.6}$$

This map decreases degrees by d because so is Ind_T.

5. Gorenstein* simplicial posets

We shall prove Theorem 1.1 in this section. Let $\mathbf{k}$ be an arbitrary field. Suppose that a simplicial poset P is Gorenstein* over $\mathbf{k}$, i.e., the order complex $\Delta(\overline{P})$ of $\overline{P} = P - \{\hat{0}\}$, which is a simplicial complex, is Gorenstein* over $\mathbf{k}$. According to Theorem II.5.1 in [9], a simplicial complex Δ of dimension $d-1$ is Gorenstein* over $\mathbf{k}$ if and only if for all $p \in |\Delta|$,

$$\widetilde{H}_q(|\Delta|,\mathbf{k}) \cong H_q(|\Delta|,|\Delta|-p;\mathbf{k}) \cong \begin{cases} \mathbf{k}, & q = d-1,\\ 0, & q < d-1.\end{cases}$$

Therefore, it follows from the universal coefficient theorem that if a simplicial poset P is Gorenstein* over $\mathbf{k}$, then it is Gorenstein* over $\mathbb{Q}$. In the sequel we may assume $\mathbf{k} = \mathbb{Q}$. According to Theorem II.5.1 in [9] again, $\Delta(\overline{P})$ is an orientable pseudomanifold, so the assumption in Section 4 is satisfied for the Gorenstein* simplicial poset P because $\Delta(\overline{P})$ is the barycentric subdivision of the CW-complex $\Gamma(P)$.

Since a Gorenstein* simplicial poset is Cohen-Macaulay, $h_i = h_i(P)$ agrees with the dimension of the homogeneous part of degree i in $A_P/(\theta_1,\dots,\theta_d)$, see the proof of Theorem 3.10 in [8]. Therefore, if $h_i = 0$ for some i ($1 \le i \le d-1$), then a product of d elements in $(A_P)_1$ vanishes, in particular, it is zero when evaluated by the index map in (4.6).

We take a subset I of $\{1,\dots,n\}$ with cardinality d such that $I = \mathcal{A}(y)$ for some $y \in P_d$. If $\mathcal{A}(y) = \mathcal{A}(y')(= I)$, then $m(y) = m(y')$ by (3.6). Therefore we may write $m(y)$ as m_I. Since

$$\iota_y(\prod_{i\in I} x_i) = \begin{cases} \prod_{i\in\mathcal{A}(y)} \theta_i(y) & \text{if } \mathcal{A}(y) = I,\\ 0 & \text{otherwise}\end{cases}$$

by Lemma 3.3, we have

$$\mathrm{Ind}_T(m_I \prod_{i\in I} x_i) = \sum_{\mathcal{A}(y)=I} \epsilon(y) \in \mathbb{Q}$$

by (4.2). Hence, if we regard $m_I \prod_{i\in I} x_i$ as an element in $A_P/(\theta_1, \dots, \theta_d)$, then we have

$$\mathrm{Ind}(m_I \prod_{i\in I} x_i) = \sum_{\mathcal{A}(y)=I} \epsilon(y).$$

Now suppose that $h_i = 0$ for some i $(1 \le i \le d-1)$. Then the left hand side at the above identity is zero as remarked above. This means that (since $\epsilon(y) = \pm 1$) there must be an even number of elements $y \in P_d$ with $\mathcal{A}(y) = I$ at the right hand side. Since I is arbitrary, we conclude that f_{d-1}(the number of elements in P_d) is even. This means that when d is even, $h_{d/2}$ must be even because $f_{d-1} = \sum_{i=0}^{d} h_i$ by (1.1) and we have Dehn-Sommerville equations $h_i = h_{d-i}$ for all i. This completes the proof of Theorem 1.1.

Remark. An element corresponding to ω_T in Section 2 is $\sum_I m_I \prod_{i\in I} x_i$ where I runs over all subsets of $\{1, \dots, n\}$ with cardinality d and m_I is understood to be zero if there is no $y \in P_d$ such that $I = \mathcal{A}(y)$.

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