COMPLETE GENERA OF CLOSED ORIENTABLE 3-MANIFOLDS

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Abstract

A complete invariant for closed connected orientable 3-manifolds is an invariant such that any two closed connected orientable 3-manifolds are homeomorphic if and only if the invariants are equal. Further, if we can reconstruct the 3-manifold from the invarinat, then it is faithfully-complete. In this paper, we construct a faithfully-complete non-negative rational invariant for closed connected orientable 3-manifolds which we call the complete genus. The value of the complete genus is greater than or equal to the Heegaard genus. This construction is made by using the faithfully-complete lattice point invariant for closed connected orientable 3-manifolds in [6]. We also construct a faithfully-complete non-negative rational invariant smaller than or equal to $\frac{1}{2}$ which we call the complete arith-genus to obtain a two-variable holomorphic function which classifies all the closed connected orientable 3-manifolds.

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1. Introduction

It is classcially well-known (cf. B. von Kerékjártó [9]) that every closed connected orientable surface F is characterized by the maximal number, say $n(\geqq 0)$ of mutually disjoint simple loops ℓ_i $(i = 1, 2, \dots, n)$ in F such that the complement $F - \cup_{i=1}^{n} \ell_i$ is connected. This number n is called the *genus* of F. We consider a union L^0 of n mutually disjoint 0-spheres in the 2-sphere S^2 (namely, a set of $2n$ points in S^2) as an S^0-*link with* n *components*. Then the surface characterization stated above is dual to the statement that F is obtained as the 1-handle surgery manifold $\chi(L^0)$ of S^2 along an S^0-link L^0 with n components for some $n \geqq 0$. Let $\mathbb{M}^2$ be the set of (the unoriented types of) closed connected orientable surfaces, and $\mathbb{L}^0$ the set of (unoriented types of) S^0-links. Since any two S^0-links with the same component number are the same link, we have a well-defined embedding

$$\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{L}^0$$

sending a surface $F \in \mathbb{M}^2$ to an S^0-link $L^0 \in \mathbb{L}^0$ such that $\chi(L^0) = F$. Further, let $\mathbb{X}^0$ be the set of non-negative integers, and $\mathbb{G}^0$ the set of (isomorphism classes of) the link groups $\pi_1(S^2 - L^0)$ of all S^0-links $L^0 \in \mathbb{L}^0$. Then we have further two natural embeddings

$$\sigma^0 : \mathbb{L}^0 \longrightarrow \mathbb{X}^0,$$
$$\pi^0 : \mathbb{L}^0 \longrightarrow \mathbb{G}^0$$

such that $\sigma^0(L^0) = n$ for an S^0-link L^0 with n components and $\pi^0(L^0) = \pi_1(S^2 - L^0)$, respectively, so that we have the composite embeddings

$$g = \sigma^0_\alpha = \sigma^0\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{X}^0,$$
$$\pi^0_\alpha = \pi^0\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{G}^0.$$

For every surface $F \in \mathbb{M}^2$, the number $g(F) = n$ is equal to the genus of F, and the group $\pi^0_\alpha(F)$ is a free group of rank $2n-1$ ($n \geqq 1$) or the trivial group $\{1\}$ ($n = 0$). Thus, the genus $g(F)$ determines the S^0-link $\alpha^0(F)$, the group $\pi^0_\alpha(F)$ and the surface F itself.

As we did in the paper [6], an analogous classification is possible for closed connected orientable 3-manifolds in place of closed connected orientable surfaces, although the existence of lots of non-trivial links in the 3-sphere S^3 makes the classification complicated. Here, for convenience we explain an idea of this classification of [6] briefly. Let $\mathbb{M}$ be the set of (unoriented types of) closed connected orientable 3-manifolds, and $\mathbb{L}$ the set of (unoriented types of) links in S^3. The *set* $\mathbb{X}$ *of lattice points* is the disjoint union of $\mathbb{Z}^n$ for all $n = 1, 2, 3, \dots$ where $\mathbb{Z}$ is the set of integers. An element $\mathbf{x} \in \mathbb{Z}^n$ is called a *lattice point* of *length* $\ell(\mathbf{x}) = n$. We consider the set $\mathbb{X}$ as a well-ordered set as it is explained in §2. Since every lattice point $\mathbf{x} \in \mathbb{X}$ induces the closed braid diagram $\mathrm{cl}\beta(\mathbf{x})$, we can introduce an equivalence relation $\sim$ in $\mathbb{X}$ so that $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathrm{cl}\beta(\mathbf{x})$ and $\mathrm{cl}\beta(\mathbf{y})$ are the same link in $\mathbb{L}$ modulo split additions of trivial links. Let $[\mathbf{x}]$ be the equivalence class of $\mathbf{x} \in \mathbb{X}$. Using this well-order of $\mathbb{X}$ and the Alexander theorem that every link is deformed into a closed braid form(cf. [1]), we can define a map

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X},$$

injective modulo split additions of trivial links, so that for every $L \in \mathbb{L}$, $\sigma(L)$ is the initial element of the equivalence class $[\mathbf{x}]$ for a lattice point $\mathbf{x} \in \mathbb{X}$ with $\mathrm{cl}(\mathbf{x}) = L$. In particular, the restriction of σ to the subset $\mathbb{L}^{\mathrm{p}}$ of prime links is injective, so that we can consider $\mathbb{L}^{\mathrm{p}}$ as a well-ordered set by the order induced from $\mathbb{X}$. The *length* $\ell(L)$ of a prime link $L \in \mathbb{L}^{\mathrm{p}}$ is the length $\ell(\sigma(L))$. Let $\mathbb{G}$ be the set of (isomorphism types of) the link groups $\pi_1(S^3 - L)$ for all links L in S^3. Let $\pi : \mathbb{L} \to \mathbb{G}$ be the map sending a link L to the link group $\pi_1(S^3 - L)$. Let $\mathbb{L}^\pi$ be the subset of $\mathbb{L}^{\mathrm{p}}$ consisting of a π-*minimal link*, that is, a prime link L which is the initial element of the subset

$$\{L' \in \mathbb{L}^{\mathrm{p}} | \pi_1(S^3 - L') = \pi_1(S^3 - L)\}$$

of $\mathbb{L}^{\mathrm{p}}$. We are interested in this subset $\mathbb{L}^{\pi}$ since it has a crucial property that the restriction of π to $\mathbb{L}^{\pi}$ is injective. Since the restriction of σ to $\mathbb{L}^{\pi}$ is also injective, we can consider $\mathbb{L}^{\pi}$ as a well-ordered set by the order induced from the order of $\mathbb{X}$. In [5], we showed that the set

$$\mathbb{L}^{\pi}(M) = \{L \in \mathbb{L}^{\pi} | \chi(L, 0) = M\}$$

is not empty for every 3-manifold $M \in \mathbb{M}$, where $\chi(L, 0)$ denotes the 0-surgery manifold of S^3 along L . By R. Kirby's theorem [10] on the Dehn surgery of framed links, we note that the set $\mathbb{L}^{\pi}(M)$ is defined in terms of only links so that any two π-minimal links in $\mathbb{L}^{\pi}(M)$ are related by two kinds of Kirby moves and choices of orientations of S^3, although the definition of $\mathbb{L}^{\pi}(M)$ above depends on homeomorphisms on 3-manifolds. Sending M to the initial link of $\mathbb{L}^{\pi}(M)$, we have an embedding

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}$$

with $\chi(\alpha(M), 0) = M$ for every 3-manifold $M \in \mathbb{M}$ which induces two embeddings

$$\sigma_{\alpha} = \sigma\alpha : \mathbb{M} \longrightarrow \mathbb{X},$$
$$\pi_{\alpha} = \pi\alpha : \mathbb{M} \longrightarrow \mathbb{G}.$$

To calculate the group $\pi_{\alpha}(M)$, we proposed a program on the classification problem (see J. Hempel [3]) of $\mathbb{M}$ and classified the prime links of lengths $\leqq 7$ and the 3-manifolds of lengths $\leqq 7$ in [6], where the *length* $\ell(M)$ of a 3-manifold $M \in \mathbb{M}$ is the length $\ell(\sigma_{\alpha}(M))$. The prime links in $\mathbb{L}^{\mathrm{p}}$ of lengths $\leqq 10$ and the 3-manifolds in $\mathbb{M}$ of lengths $\leqq 10$ will be enumerated in [7] and [8], respectively, where our enumeration of $\mathbb{L}^{\mathrm{p}}$ is based on the well-order of $\mathbb{X}$ and differtent from the enumerations of J. H. Conway [2] and D. Rolfsen [13].

By a special featur of the 0-surgery, the S^0-link $\alpha(M) \cap S^2$ in S^2 produces a surface $\chi(\alpha(M) \cap S^2)$ naturally embedded in M with $\alpha^0(\chi(\alpha(M) \cap S^2)) = \alpha(M) \cap S^2$ for every 2-sphere S^2 in S^3 meeting the link $\alpha(M)$ transversely. In this sense, the embedding α is an extension of the embedding α^0. In this construction, we can reconstruct the link $\alpha(M)$, the group $\pi_{\alpha}(M)$ and the 3-manifold M itself from the lattice point $\sigma_{\alpha}(M) \in \mathbb{X}$, and thus we have constructed the embeddings α, σ_{α}, and π_{α} analogous to the embeddings α^0, σ_{α}^0, and π_{α}^0, respectively. In general, we say that an invariant $I(M)$ of a 3-manifold $M \in \mathbb{M}$ taking a value in an algebraic system is *complete* if $I(M) = I(M')$ means $M = M'$ in $\mathbb{M}$. The complete invariant $I(M)$ is *faithfully-complete* if M can be reconstructed from the data of $I(M)$. For example, $\pi_{\alpha}(M)$ is a complete invariant taking the value in groups and $\sigma_{\alpha}(M)$ is a faithfully-complete invariant taking the value in lattice points.

The main work of this paper is to propose a faithfully-complete invariant $g(M)$ taking the value in the set $\mathbb{Q}_{0+}$ of non-negative rational numbers and called the *complete genus* of M by using the complete invariant $\sigma_{\alpha}(M)$. For this purpose, we embed a subset Δ^{+} of $\mathbb{X}$ containing the image $\sigma(\mathbb{L}^{\mathrm{p}})$ into the set $\mathbb{Q}_{0+}$ in §3. This set Δ^+ is a natural generalization of the delta set Δ defined in [6], as it is explained in §2. In §4, we shall give a table of complete genera of all the 3-manifolds with lengths $\leqq 7$ together with some extra data for convenience. Some properties of this complete genus are as follows (see §5 for more detailed properties):

Properties of the complete genus.

(1) We have an equality $g(M) \geqq h(M)$ for every $M \in \mathbb{M}$, where $h(M)$ denotes the Heegaard genus of $M \in \mathbb{M}$.
(2) $g(S^3) = 0$, $g(S^1 \times S^2) = 1$ and $g(M)$ belongs to the open interval $(n - \frac{1}{2}, n)$ or $(n, n + \frac{1}{2})$ with $n \geqq 3$ for every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$.
(3) For every integer $n \geqq 3$, there are only finitely many 3-manifolds $M \in \mathbb{M}$ such that $g(M)$ belongs to $(n - \frac{1}{2}, n + \frac{1}{2})$.
(4) From the value of $g(M)$, we can reconstruct the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself.

Since for every positive integer n, there are infinitely many 3-manifolds $M \in \mathbb{M}$ with $h(M) = n$, we see from the property (2) that the inequality of (3) must be far from the equality in general. In §6, as a variant of the complete genus $g(M)$, we shall propose a related faithfully-complete invariant $a(M)$ of a 3-manifold $M \in \mathbb{M}$ which takes the value in non-negative rational numbers smaller than or equal to $\frac{1}{2}$ and is called the *complete arith-genus* of M. The following properties of this complete arith-genus are given in §6:

Properties of the complete arith-genus.

(1) $a(S^3) = 0$, $a(S^1 \times S^2) = \frac{1}{2}$ and $0 < a(M) < \frac{1}{2}$ for every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$.
(2) We can reconstruct from the value of $a(M)$ the complete genus $g(M)$, the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself.

The list of complete arith-genera of all the 3-manifolds with lengths $\leqq 7$ is also made in §6. A reason why we introduce this complete arith-genus is because *using it, we can construct a holomorphic function $\mu(u, z)$ with $\mathbb{C}^2$ the absolute convergence domain which classifies all the 3-manifolds $M \in \mathbb{M}$*. This construction is also done in §6.

We mention here some analogous invariants derived from different viewpoints: J. Milnor and W. Thurston defined in [11] non-negative real-valued invariants of closed connected 3-manifolds by the property that if $\tilde{N} \to N$ is a degree $n(\geqq 2)$ connected covering of a closed connected 3-manifold N, then the invariant of $\tilde{N}$ is n times the invariant of N. Since by this property any two homotopy equivalent, non-homeomorphic lens spaces must have the same invariant, we see that these invariants are not any complete invariants. Also, Y. Nakagawa defined in [12] a family of integer-valued complete invariants of the set of knots by using R. W.

Ghrist's universal template, although at present the computation of Nakagawa's invariants and a generalization to $\mathbb{L}$ appear difficult.

2. The range of prime links in the set of lattice points

In this section, we are interested in a subset of $\mathbb{X}$ containing the image $\sigma(\mathbb{L}^{\mathrm{p}})$. For this purpose, we need some notations on lattice points given in [6]. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of length n, we denote the lattice points $(x_n, \ldots, x_2, x_1)$ and $(|x_1|, |x_2|, \ldots, |x_n|)$ by $\mathbf{x}^T$ and $|\mathbf{x}|$, respectively. Let $|\mathbf{x}|_p$ be a permutation $(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ of the coordinates $|x_j|$ $(j = 1, 2, \ldots, n)$ of $|\mathbf{x}|$ such that

$$|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|.$$

Let $\min |\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i|$ and $\max |\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|$. The *dual* lattice point $\delta(\mathbf{x}) = (x'_1, x'_2, \ldots, x'_n)$ of $\mathbf{x}$ is defined by

$$x'_i = \begin{cases} \mathrm{sign}(x_i)(\max |\mathbf{x}| + 1 - |x_i|) & x_i \neq 0 \\ 0 & x_i = 0. \end{cases}$$

Defining $\delta^0(\mathbf{x}) = \mathbf{x}$ and $\delta^n(\mathbf{x}) = \delta(\delta^{n-1}(\mathbf{x}))$ inductively, we note that $\delta^2(\mathbf{x}) \neq \mathbf{x}$ in general, but $\delta^{n+2}(\mathbf{x}) = \delta^n(\mathbf{x})$ for all $n \geqq 1$. For a lattice point $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ of length m, we denote by $(\mathbf{x}, \mathbf{y})$ the lattice point

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

of length $n + m$. For an integer m and a positive integer n, we denote by m^n the lattice point $(m, m, \ldots, m)$ of length n. Also, we take $-m^n = (-m)^n$. The *canonical order* of $\mathbb{X}$ is a well-order determined as follows: Namely, the well-order in $\mathbb{Z}$ is defined by

$$0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$$

(understood as an order counted on the real line along a spiral curve in the complex number plane starting from the origin and rounding counterclockwise). This order of $\mathbb{Z}$ is extended to a well-order in $\mathbb{Z}^n$ for every $n \geqq 2$ as follows: Namely, for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^n$ we define $\mathbf{x}_1 < \mathbf{x}_2$ if we have one of the following conditions (1)-(3):

(1) $|\mathbf{x}_1|_p < |\mathbf{x}_2|_p$ by the lexicographic order (on the natural number order).
(2) $|\mathbf{x}_1|_p = |\mathbf{x}_2|_p$ and $|\mathbf{x}_1| < |\mathbf{x}_2|$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}_1| = |\mathbf{x}_2|$ and $\mathbf{x}_1 < \mathbf{x}_2$ by the lexicographic order on the well-order of $\mathbb{Z}$ defined above.

Finally, for any two lattice points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$ with $\ell(\mathbf{x}_1) < \ell(\mathbf{x}_2)$, we define $\mathbf{x}_1 < \mathbf{x}_2$.

The following subset of $\mathbb{X}$ is defined in [6]:

Definition 2.1. The *delta set* Δ is the subset of $\mathbb{X}$ consisting of $0 (\in \mathbb{Z})$, 1^n $(n \geqq 2)$ and all the lattice points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ $(n \geqq 2)$ with the following conditions (1)-(8) simultaneously:

(1) $x_1 = 1$, $|x_n| \geqq 2$ and $n/2 \geqq \max|\mathbf{x}| \geqq 2$.
(2) For every integer k such that $1 < k < \max|\mathbf{x}|$, there is an index i with $|x_i| = k$.
(3) For every index s, there is no index t $(s \leqq t)$ such that $|x_i| = |x_s|$ if and only if $s \leqq i \leqq t$.
(4) If $|x_i| > |x_{i+1}|$, then $|x_i| - 1 = |x_{i+1}|$.
(5) If $|x_i| = |x_{i+1}|$, then $\mathrm{sign}(x_i) = \mathrm{sign}(x_{i+1})$.
(6) If $\mathbf{x}$ is written as $(\mathbf{x}', \mathbf{y}, \mathbf{x}'')$ where $|\mathbf{y}|$ is $(k, (k+1)^m, k)$, $(k^m, k+1, k)$ or $(k, k+1, k^m)$ for some $k, m \geqq 1$ and neither $|\mathbf{x}'|$ nor $|\mathbf{x}''|$ has k as any coordinate, then $\mathbf{y}$ is equal to $\pm(k, -\varepsilon(k+1)^m, k)$, $\pm(\varepsilon k^m, -(k+1), k)$ or $\pm(k, -(k+1), \varepsilon k^m)$ for some $\varepsilon = \pm 1$, respectively. In particular, if $m = 1$, then we have $\varepsilon = 1$.
(7) If $\mathbf{x}$ is written as $(\mathbf{x}', \mathbf{y}, \mathbf{x}'')$ where $|\mathbf{y}| = (k+1, k^m, k+1)$ for some $k, m \geqq 1$, then $\mathbf{y} = \pm(k+1, \varepsilon k^m, k+1)$ for some $\varepsilon = \pm 1$. Further if $m = 1$, then we have $\varepsilon = -1$.
(8) $\mathbf{x}$ is the initial element (in the canonical order) of the set of the lattice points obtained from every lattice point of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$, $\pm\delta(\mathbf{x})$ and $\pm\delta(\mathbf{x})^T$ by cyclically permuting the coordinates.

The following lemma is also proved in [6]:

Lemma 2.2. The image $\sigma(\mathbb{L}^{\mathrm{p}})(\subset \mathbb{X})$ is contained in Δ.

Remark 2.3. If we can replace Δ by a smaller subset of $\mathbb{X}$ containing $\sigma(\mathbb{L}^{\mathrm{p}})$, our classification arguments on $\mathbb{L}^{\mathrm{p}}$ and $\mathbb{M}$ would be simpler. For example, let Δ^* be the subset of $\mathbb{X}$ obtained from Δ by replacing the condition (3) by the following condition (which is more restrictive):

(3^*) Any cyclic permutation of the coordinates of $\mathbf{x}$ is not written as $(\mathbf{x}', \mathbf{x}'')$ with $\max|\mathbf{x}'| < \min|\mathbf{x}''|$.

Then we still have $\sigma(\mathbb{L}^{\mathrm{p}}) \subset \Delta^* \subset \Delta$ (see [7]).

The *extended delta set* is a subset Δ^+ of $\mathbb{X}$ consisting of the zero $0 \in \mathbb{Z}$ (which is a lattice point of length 1) and all the lattice points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{X}$ of lengths $n \geqq 2$ such that $x_1 = 1$ and $|x_i| \leqq n/2$ $(i = 2, 3, \ldots, n)$. By definition, we have

$$\sigma(\mathbb{L}^{\mathrm{p}}) \subset \Delta \subset \Delta^+.$$

For most arguments of this paper, we shall discuss the extended delta set Δ^+ rather than Δ.

3. Embedding the extended delta set into the set of non-negative rational numbers

For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta^+$, we define the rational number

$$\zeta(\mathbf{x}) = n + \frac{x_2}{(n+1)^{n-1}} + \cdots + \frac{x_n}{n+1}.$$

We note that $\zeta(0) = 1$. For a rational number r, let $\mathrm{int}(r)$ denote the maximal integer which does not exceed r. We show the following lemma:

Lemma 3.1. The map $\mathbf{x} \mapsto \zeta(\mathbf{x})$ induces an embedding

$$\zeta : \Delta^+ \longrightarrow \mathbb{Q}_{0+}$$

such that for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta^+$ with $n \geqq 2$ we have the following properties (1)-(3):

(1) For $n \geqq k+1 \geqq 2$, let $\varepsilon_{n,k}$ be the rational number defined by

$$\varepsilon_{n,k} = \begin{cases} \zeta(\mathbf{x}) - n & (k=1) \\ \zeta(\mathbf{x}) - n - (\frac{x_{n-k+2}}{(n+1)^{k-1}} + \cdots + \frac{x_n}{n+1}) & (k \geqq 2). \end{cases}$$

Then we have

$$|\varepsilon_{n,k}| \leqq \frac{1}{2(n+1)^{k-1}} - \frac{1}{2(n+1)^{n-1}},$$

where the equality holds if and only if $x_2 = \cdots = x_{n-k+1} = \pm\frac{n}{2}$.

(2) We have $|\zeta(\mathbf{x}) - n| < \frac{1}{2}$ and $\mathrm{int}(\zeta(\mathbf{x}) + \frac{1}{2}) = n$.

(3) We can reconstruct the lattice point $\mathbf{x} \in \Delta^+$ from the value of $\zeta(\mathbf{x})$.

Further, for any integer $n \geqq 2$, there are only finitely many $\mathbf{x} \in \Delta^+$ such that $\zeta(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})$.

Proof. To show (1), we note that $|x_i| \leqq n/2$ for all i. Then we have

$$\begin{aligned} |\varepsilon_{n,k}| &= |\frac{x_2}{(n+1)^{n-1}} + \cdots + \frac{x_{n-k+1}}{(n+1)^k}| \\ &\leqq \frac{n}{2}(\frac{1}{(n+1)^{n-1}} + \cdots + \frac{1}{(n+1)^k}) \\ &= \frac{n}{2} \cdot \frac{1}{(n+1)^{n-1}} \left(1 + (n+1) + \cdots + (n+1)^{n-k-1}\right) \\ &= \frac{n}{2} \cdot \frac{1}{(n+1)^{n-1}} \cdot \frac{(n+1)^{n-k} - 1}{n} \\ &= \frac{1}{2(n+1)^{k-1}} - \frac{1}{2(n+1)^{n-1}}. \end{aligned}$$

This inequality $\leqq$ is replaced by the equality $=$ if and only if $x_2 = \cdots = x_{n-k+1} = \pm\frac{n}{2}$, showing (1). To show (2), we note by (1) taking $k = 1$ that

$$|\zeta(\mathbf{x}) - n| \leqq \frac{1}{2} - \frac{1}{2(n+1)^{n-1}} < \frac{1}{2}.$$

This also implies that

$$\mathrm{int}(\zeta(\mathbf{x}) + \frac{1}{2}) = n.$$

To show that ζ is an embedding, we note that $\zeta(\mathbf{x}) \neq \zeta(0) = 1$ and that if the value of $\zeta(\mathbf{x})$ is given, then the length $n(\geqq 2)$ of $\mathbf{x}$ is determined uniquely. We assume

$$\zeta(\mathbf{x}) - n = \frac{x_2'}{(n+1)^{n-1}} + \cdots + \frac{x_n'}{n+1}$$

for arbitrary integres x_i' with $|x_i'| \leqq n/2$ $(i = 2, 3, \ldots, n)$. Then inductively we have

$$x_i' - x_i \equiv 0 \pmod{n+1} \quad (i = 2, 3, \ldots, n).$$

Since

$$|x_i' - x_i| \leqq |x_i'| + |x_i| \leqq \frac{n}{2} + \frac{n}{2} = n,$$

we must have $x_i' - x_i = 0 (i = 2, 3, \ldots, n)$. Since $x_1 = 1$ for $n \geqq 2$, we see that the map $\zeta : \Delta^+ \to \mathbb{Q}_{0+}$ is injective. To see (3), we note that x_2 is determined by $|x_2| \leqq \frac{n}{2}$ and

$$x_2 \equiv (n+1)^{n-1}(\zeta(\mathbf{x}) - n) \pmod{n+1}.$$

Similarly, x_k $(k \geqq 3)$ is inductively determined by $|x_k| \leqq \frac{n}{2}$ and

$$x_k \equiv (n+1)^{n-k+1}(\zeta(\mathbf{x}) - n - \frac{x_2}{(n+1)^{n-1}} - \cdots - \frac{x_{k-1}}{(n+1)^{n-k+2}}) \pmod{n+1},$$

showing (3). Finally, if $\zeta(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})$, then we have $\mathrm{int}(\zeta(\mathbf{x}) + \frac{1}{2}) = n$ and the length n of $\mathbf{x}$ is determined. Since there are only finitely many lattice points of length n in Δ^+, there are only finitely many lattice points $\mathbf{x} \in \Delta^+$ with $\zeta(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})$. $\square$

4. The table of genera of 3-manifolds with up to 7 lengths

By the classification of [6], if $\ell(M) = 1, 2$, then we have $M = S^1 \times S^2$, S^3, respectively. The reason on $\ell(M) = 2$ is because we take the 3-sphere S^3 as the 0-surgery manifold of S^3 along the Hopf link 2_1^2, so that we have $\sigma_\alpha(S^3) = 1^2$. However, *we can also take S^3 as the 3-manifold without doing the 0-surgery of S^3 along a link.* In this paper, we adopt this viewpoint and introduce the empty lattice point $\boldsymbol{\emptyset} \in \Delta \subset \Delta^+ \subset \mathbb{X}$ of length 0, the empty link $\emptyset \in \mathbb{L}$ to define

$$\alpha(S^3) = \emptyset, \quad \sigma_\alpha(S^3) = \boldsymbol{\emptyset}, \quad \zeta(\boldsymbol{\emptyset}) = 0.$$

Also, we have $\pi_\alpha(S^3) = \{1\}$ by introducing the trivial group $\{1\}$ to the set $\mathbb{G}$ of link groups. In this viewpoint, we note that *there is no* 3*-manifold* $M \in \mathbb{M}$ *with* $\ell(M) = 2$. Then we make the following definition of complete genus:

Definition 4.1. The *complete genus* $g(M)$ of a 3-manifold $M \in \mathbb{M}$ is given by the identity

$$g(M) = \zeta(\sigma_\alpha(M)).$$

It is direct that $g(S^3) = 0$ and $g(S^1 \times S^2) = 1$. We shall confirm in Theorem 5.2 that $g(M)$ is a faithfully-complete invariant. Since $\sigma_\alpha(\mathbb{M}) \subset \Delta$ and every initial segment of Δ is a finite set, there are only finitely many 3-manifolds with length n for every $n \geqq 0$. According to the canonical well-order of $\mathbb{X}$, we enumerate the 3-manifolds of length n as follows:

$$M_{n,1} < M_{n,2} < \cdots < M_{n,m_n}$$

for a non-negative integer m_n depending only on n. Let $\mathbf{x}_{n,i} = \sigma_\alpha(M_{n,i}) \in \Delta$ and $g_{n,i} = g(M_{n,i})$. By [6], we reconstruct from $\mathbf{x}_{n,i}$ the link $\alpha(M_{n,i}) = L_{n,i} \in \mathbb{L}$, the group $\pi_\alpha(M_{n,i}) = G_{n,i} \in \mathbb{G}$ and the 3-manifold $M_{n,i}$ itself. By (1) of Lemma 3.1, we reconstruct $\mathbf{x}_{n,i}$ from $g_{n,i}$, so that we can construct from $g_{n,i}$ the lattice point $\mathbf{x}_{n,i}$, the link $L_{n,i}$, the group $G_{n,i}$ and the 3-manifold $M_{n,i}$ itself.

In [6], we listed all the lattice points $\mathbf{x}_{n,i}$ and the links $L_{n,i}$ identified with the notations in D. Rolfsen's table [13] for all $n \leqq 7$. In the following table, we list all the genera $g_{n,i}$ together with $\mathbf{x}_{n,i}$ and$L_{n,i}$ for all $n \leqq 7$, where we note that there is no 3-manifold of length 2 by the reason stated above (different from the list of [6] at this point). For convenience, we also include the homological data in this table by letting $H_{n,i} = H_1(M_{n,i}; \mathbb{Z})$.

Table 4.2.

$g_{0,1} = 0$, $\mathbf{x}_{0,1} = \boldsymbol{\emptyset}$, $L_{0,1} = \emptyset$, $H_{0,1} = 0$.

$g_{1,1} = 1$, $\mathbf{x}_{1,1} = 0$, $L_{1,1} = O$, $H_{1,1} = \mathbb{Z}$.

$g_{3,1} = 3 + \frac{5}{4^2} = 3.3125$, $\mathbf{x}_{3,1} = 1^3$, $L_{3,1} = 3_1$, $H_{3,1} = \mathbb{Z}$.

$g_{4,1} = 4 + \frac{31}{5^3} = 4.248$, $\mathbf{x}_{4,1} = 1^4$, $L_{4,1} = 4_1^2$, $H_{4,1} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$g_{4,2} = 4 - \frac{47}{5^3} = 3.624$, $\mathbf{x}_{4,2} = (1, -2, 1, -2)$, $L_{4,2} = 4_1$, $H_{4,2} = \mathbb{Z}$.

$g_{5,1} = 5 + \frac{259}{6^4} = 5.199845679...$, $\mathbf{x}_{5,1} = 1^5$, $L_{5,1} = 5_1$, $H_{5,1} = \mathbb{Z}$.
$g_{5,2} = 5 - \frac{407}{6^4} = 4.685956790...$, $\mathbf{x}_{5,2} = (1^2, -2, 1, -2)$, $L_{5,2} = 5_1^2$, $H_{5,2} = \mathbb{Z} \oplus \mathbb{Z}$.

$g_{6,1} = 6 + \frac{2801}{7^5} = 6.16665675...$, $\mathbf{x}_{6,1} = 1^6$, $L_{6,1} = 6_1^2$, $H_{6,1} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
$g_{6,2} = 6 + \frac{4565}{7^5} = 6.271613018...$, $\mathbf{x}_{6,2} = (1^3, 2, -1, 2)$, $L_{6,2} = 5_2$, $H_{6,2} = \mathbb{Z}$.

$g_{6,3} = 6 - \frac{4549}{7^5} = 5.729338965...$, $\mathbf{x}_{6,3} = (1^3, -2, 1, -2)$, $L_{6,3} = 6_2$, $H_{6,3} = \mathbb{Z}$.
$g_{6,4} = 6 + \frac{5209}{7^5} = 6.309930386...$, $\mathbf{x}_{6,4} = (1^2, 2, 1^2, 2)$, $L_{6,4} = 6^3_3$, $H_{6,4} = \mathbb{Z}_2$.
$g_{6,5} = 6 - \frac{4423}{7^5} = 5.736835842...$, $\mathbf{x}_{6,5} = (1^2, -2, 1^2, -2)$, $L_{6,5} = 6^3_1$, $H_{6,5} = \mathbb{Z}_2$.
$g_{6,6} = 6 - \frac{5452}{7^5} = 5.675611352...$, $\mathbf{x}_{6,6} = (1^2, -2, 1, -2^2)$, $L_{6,6} = 6_3$, $H_{6,6} = \mathbb{Z}$.
$g_{6,7} = 6 - \frac{4552}{7^5} = 5.729160468...$, $\mathbf{x}_{6,7} = (1, -2, 1, -2, 1, -2)$, $L_{6,7} = 6^3_2$, $H_{6,7} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.
$g_{6,8} = 6 + \frac{6669}{7^5} = 6.396798952...$, $\mathbf{x}_{6,8} = (1, -2, 1, 3, -2, 3)$, $L_{6,8} = 6^2_3$, $H_{6,8} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$g_{7,1} = 7 + \frac{37449}{8^6} = 7.142856597...$, $\mathbf{x}_{7,1} = 1^7$, $L_{7,1} = 7_1$, $H_{7,1} = \mathbb{Z}$.
$g_{7,2} = 7 + \frac{62537}{8^6} = 7.238559722...$, $\mathbf{x}_{7,2} = (1^4, 2, -1, 2)$, $L_{7,2} = 6^2_2$, $H_{7,2} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
$g_{7,3} = 7 - \frac{62391}{8^6} = 6.761997222...$, $\mathbf{x}_{7,3} = (1^4, -2, 1, -2)$, $L_{7,3} = 7^2_1$, $H_{7,3} = 0$.
$g_{7,4} = 7 - \frac{61047}{8^6} = 6.767124176...$, $\mathbf{x}_{7,4} = (1^3, -2, 1^2, -2)$, $L_{7,4} = 7^2_4$, $H_{7,4} = \mathbb{Z} \oplus \mathbb{Z}$.
$g_{7,5} = 7 - \frac{73335}{8^6} = 6.720249176...$, $\mathbf{x}_{7,5} = (1^3, -2, 1, -2^2)$, $L_{7,5} = 7^2_2$, $H_{7,5} = 0$.
$g_{7,6} = 7 - \frac{73167}{8^6} = 6.720890045...$, $\mathbf{x}_{7,6} = (1^2, -2, 1^2, -2^2)$, $L_{7,6} = 7^2_5$, $H_{7,6} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$g_{7,7} = 7 - \frac{62415}{8^6} = 6.761905670...$, $\mathbf{x}_{7,7} = (1^2, -2, 1, -2, 1, -2)$, $L_{7,7} = 7^2_6$, $H_{7,7} = \mathbb{Z} \oplus \mathbb{Z}$.
$g_{7,8} = 7 - \frac{91695}{8^6} = 6.650211334...$, $\mathbf{x}_{7,8} = (1^2, 2, -1, -3, 2, -3)$, $L_{7,8} = 6_1$, $H_{7,8} = \mathbb{Z}$.
$g_{7,9} = 7 + \frac{91697}{8^6} = 7.349796295...$, $\mathbf{x}_{7,9} = (1^2, -2, 1, 3, -2, 3)$, $L_{7,9} = 7_6$, $H_{7,9} = \mathbb{Z}$.
$g_{7,10} = 7 + \frac{91526}{8^6} = 7.349143981...$, $\mathbf{x}_{7,10} = (1, -2, 1, -2, 3, -2, 3)$, $L_{7,10} = 7_7$, $H_{7,10} = \mathbb{Z}$.
$g_{7,11} = 7 + \frac{89286}{8^6} = 7.34059906...$, $\mathbf{x}_{7,11} = (1, -2, 1, 3, -2^2, 3)$, $L_{7,11} = 7^3_1$, $H_{7,11} = \mathbb{Z}_2$.

5. Properties of the complete genus

For every 3-manifold $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^3$, we have $n = \ell(M) \geqq 3$. Let x_n be the nth coordinate of the lattice point $\sigma_\alpha(M) \in \Delta$. By the definition of Δ, we have $x_n \neq 0$. We say that the 3-manifold M is *positive* or *negative*, respectively, according to if $x_n > 0$ or $x_n < 0$. Every 3-manifold $M \in \mathbb{M}$ is obtained from two handlebodies by pasting along the boundaries which is referred to as a *Heegaard splitting* of M. The *Heegaard genus*, $h(M)$ of M is the minimum of the genera of such handlebodies. The following relationship between a bridge presentation of a link $L \in \mathbb{L}$(see [4]) and Heegaard splittings of the Dehn surgery manifolds along L is a folk result (although we could not find the proof).

Lemma 5.1. Let a link $L \in \mathbb{L}$ have a g-bridge presentation. Then every Dehn surgery manifold M along L admits a Heegaard splitting of genus g.

Proof. Since S^3 is a union of two 3-balls B, B' pasting along the boundary spheres such that $T = L \cap B$ and $T' = L \cap B'$ are trivial tangles of g proper arcs in B and B', respectively. Let $N(T)$ be a tubular neighborhood of T in B, $V = \mathrm{cl}(B - N(T))$, and $V' = B' \cup N(T)$. By construction, V and V' are handlebodies of genus g and forms a Heegaard splitting of S^3. To complete the proof, it suffices to show that the Dehn surgery from S^3 to M along L just changes V' into another handlebody V'', so that V and V'' forms a Heegaard splitting of M of genus g. Since T' is a trivial tange in B' of g proper arcs, there are $g-1$ proper disks D_i $(i = 1, 2, \ldots, g-1)$ in B' which split B' into a 3-manifold regarded as a tubular neighborhood $N(T')$ of T' in B'. Then the union $N(L) = N(T) \cup N(T')$ is regarded as a tubular neighborhood of L in S^3. The Dehn surgery from S^3 to M along L just changes $N(L)$ into another union of solid tori without changing the boundary $\partial N(L)$. Thus, we obtain a desired handlebody V'' by pasting along the disks corresponding to D_i $(i = 1, 2, \ldots, g-1)$. □

By convention, we have $\max|\emptyset| = -1$. We show the following theorem:

Theorem 5.2. The complete genus $g(M)$ of every $M \in \mathbb{M}$ is a faithfully-complete invariant such that we have

$$h(S^3) = g(S^3) = \ell(S^3) = 0,$$
$$h(S^1 \times S^2) = g(S^1 \times S^2) = \ell(S^1 \times S^2) = 1$$

and the following properties for every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$:

(1) We have
$$h(M) \leqq \max|\sigma_\alpha(M)| + 1 \leqq \frac{\ell(M)}{2} + 1.$$

(2) According to whether M is positive or negative, we have
$$n + \frac{1}{2(n+1)} + \frac{1}{2(n+1)^n} < g(M) < n + \frac{1}{2} - \frac{1}{2(n+1)^n} \quad \text{or}$$
$$n - \frac{1}{2} + \frac{1}{2(n+1)^n} < g(M) < n - \frac{3}{2(n+1)} - \frac{1}{2(n+1)^n},$$
respectively, where $n = \ell(M) (\geqq 3)$.

(3) For every integer $n \geqq 3$, there are only finitely many 3-manifolds $M \in \mathbb{M}$ such that $g(M)$ belongs to the open interval $(n - \frac{1}{2}, n + \frac{1}{2})$.

(4) We can reconstruct not only the 3-manifold M itself but also the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$, the group $\pi_\alpha(M)$ from the value of $g(M)$.

Proof. By definition, we have $h(S^3) = g(S^3) = \ell(S^3) = 0$ and $h(S^1 \times S^2) = g(S^1 \times S^2) = \ell(S^1 \times S^2) = 1$. By the property of σ_α in [6] and Lemma 3.1, we see that $g(M)$ is a faithfully-complete invariant and the properties (3) and (4) hold. Since the link $\alpha(M)$ has a $(\max|\sigma_\alpha(M)|+1)$-bridge presentation, we see from Lemma 5.1 that
$$h(M) \leqq \max|\sigma_\alpha(M)| + 1 \leqq \frac{n}{2} + 1,$$

showing (1). We show (2). By (1) of Lemma 3.1, we have

$$\varepsilon_{n,2} = g(M) - n - \frac{x_n}{n+1}$$

and

$$|\varepsilon_{n,2}| < \frac{1}{2(n+1)} - \frac{1}{2(n+1)^{n-1}},$$

for some x_i $(2 \leqq i \leqq n-1)$ is 1 and hence not to equal to $\pm\frac{n}{2}$. Hence we have

$$n + \frac{2x_n - 1}{2(n+1)} + \frac{1}{2(n+1)^{n-1}} < g(M) < n + \frac{2x_n + 1}{2(n+1)} - \frac{1}{2(n+1)^{n-1}}.$$

When M is positive, we have $2 \leqq 2x_n \leqq n$ by a property of Δ and hence

$$n + \frac{1}{2(n+1)} + \frac{1}{2(n+1)^{n-1}} < g(M) < n + \frac{1}{2} - \frac{1}{2(n+1)^n}.$$

When M is negative, we have $-n \leqq 2x_n \leqq -4$ by the property of Δ and hence

$$n - \frac{1}{2} + \frac{1}{2(n+1)^n} < g(M) < n - \frac{3}{2(n+1)} - \frac{1}{2(n+1)^n}. \qquad \square$$

Here are some examples.

Example 5.3. (1) Let $M = \chi(3_1, 0) = M_{3,1}$ for the trefoil knot 3_1. Since the bridge index of 3_1 is 2 and M is not any lens space, we see from Table 4.2 that

$$h(M) = \max|\sigma_\alpha(M)| + 1 = 2 < \frac{\ell(M)}{2} + 1 = 2.5 \text{ and } g(M) = 3 + \frac{5}{4^2} = 3.3125.$$

(2) Let $M = \chi(4_1^2, 0) = M_{4,1}$ for the $(2,4)$-torus link 4_1^2. Since the bridge index of 4_1^2 is 2 and the first homology $H_1(M)$ has exactly 2 generators, we see from Table 4.2 that

$$h(M) = \max|\sigma_\alpha(M)| + 1 = 2 < \frac{\ell(M)}{2} + 1 = 3 \text{ and } g(M) = 4 + \frac{31}{5^3} = 4.248.$$

(3) Let $M = \chi(4_1, 0) = M_{4,2}$ for the figure eight knot 4_1. Since the bridge index of 4_1 is 2 and M is not any lens space, we see from Table 4.2 that

$$h(M) = 2, \ \max|\sigma_\alpha(M)| + 1 = 3 = \frac{\ell(M)}{2} + 1 \text{ and } g(M) = 4 - \frac{47}{5^3} = 3.624.$$

The following corollary follows from Theorem 5.2 and Example 5.3.

Corollary 5.4. For every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$, we have

(1) $\ell(M) < g(M) < \ell(M) + \frac{1}{2}$ if M is positive,
(2) $\ell(M) - \frac{1}{2} < g(M) < \ell(M)$ if M is negative,
(3) $\ell(M) = \mathrm{int}(g(M) + \frac{1}{2})$,
(4) $h(M) + 1 < g(M)$.

Proof. (1) and (2) follows from (2) of Theorem 5.2. (3) is direct from (1) and (2). To see (4), Example 5.3 shows that the inequality $h(M)+1<g(M)$ holds for $\ell(M)<5$. Let $\ell(M)\geqq 5$. If M be positive, then we have

$$g(M)>\ell(M)=(\frac{\ell(M)}{2}+1)+1+(\frac{\ell(M)}{2}-2)>h(M)+1,$$

for $\frac{\ell(M)}{2}+1\geqq h(M)$ by (1) of Theorem 5.2 and that $\ell(M)\geqq 5$. If M is negative and $\ell(M)$ is odd, then we write $\ell(M)=2s+1$ for $s\geqq 2$. Since $h(M)\leqq s+1$ and $g(M)+\frac{1}{2}>2s+1$, we have

$$g(M)\geqq 2s+\frac{1}{2}=(s+1)+1+(s-\frac{3}{2})>h(M)+1.$$

If M is negative and $\ell(M)$ is even, then we write $\ell(M)=2s$ for $s\geqq 3$. Since $h(M)\leqq s+1$ and $g(M)+\frac{1}{2}>2s$, we have

$$g(M)\geqq 2s-\frac{1}{2}=(s+1)+1+(s-\frac{5}{2})>h(M)+1. \quad \square$$

6. A holomorphic function classifying all the closed connected orientable 3-manifolds

For a lattice point $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\Delta^+$of length $n>1$, we define the rational number

$$\xi(\mathbf{x})=\left|\frac{x_1}{(n+1)^n}+\frac{x_2}{(n+1)^{n-1}}+\cdots+\frac{x_n}{n+1}\right|.$$

In other words, $\xi(\mathbf{x})$ is the absolute value of the rational number obtained from $\zeta(\mathbf{x})$ by replacing the summand n with $\frac{x_1}{(n+1)^n}$. We define

$$\xi(\boldsymbol{\emptyset})=0 \quad \text{and} \quad \xi(0)=\frac{1}{2}.$$

A reason why we adopt this last identity comes from a geometric reason that the trivial knot O is also represented by the closed braid $\mathrm{cl}\beta(1)$ and we have $\xi(1)=\frac{1}{2}$. Let $[0,\frac{1}{2}]_{\mathbb{Q}}=[0,\frac{1}{2}]\cap\mathbb{Q}$. Then we show the following lemma:

Lemma 6.1. The map $\mathbf{x}\mapsto\xi(\mathbf{x})$ induces an embedding

$$\xi:\Delta^+\longrightarrow[0,\frac{1}{2}]_{\mathbb{Q}}$$

such that

(1) we have $0<\xi(\mathbf{x})<\frac{1}{2}$ for every $\mathbf{x}\in\Delta^+$ with $\mathbf{x}\neq\boldsymbol{\emptyset},0$, and
(2) we can reconstruct the lattice point $\mathbf{x}\in\Delta^+$ from the value of $\xi(\mathbf{x})$.

Proof. Let $\mathbf{x}(\neq \emptyset, 0)$ be in Δ^+. By the proof of (2) of Lemma 3.1, we see that $\xi(\mathbf{x}) < \frac{1}{2}$. Let $\xi(\mathbf{x}) = \frac{m}{(n+1)^n}$. Since $x_1 = 1$, we see that $m \equiv 1 \pmod{(n+1)}$, so that $(m, (n+1)^n) = 1$. In particular, we have $\xi(\mathbf{x}) \neq 0$, and thus, $0 < \xi(\mathbf{x}) < \frac{1}{2}$, showing (1). To see (2), it suffices to show that $\mathbf{x}$ is reconstructed from the value of $\xi(\mathbf{x})$ uniquely. If we write $\xi(\mathbf{x}) = \frac{m}{N}$ with $(m, N) = 1$, then we have $N = (n+1)^n$ for $n = \ell(\mathbf{x})$. By applying the argument of Lemma 3.1 to $\xi(\mathbf{x})$, we find a unique lattice point $\mathbf{y} \in \mathbb{X}$ with $\xi(\mathbf{y}) = \xi(\mathbf{x})$ and $\mathbf{y} = \pm\mathbf{x}$. Using $x_1 = 1$, we have the unique lattice point $\mathbf{x} \in \Delta^+$. $\square$

The *complete arith-genus* of a 3-manifold $M \in \mathbb{M}$ is the rational number

$$a(M) = \xi(\sigma_\alpha(M)).$$

By Lemma 6.1, we have the following properties on the complete arith-genus:

(1) We have $a(S^3) = 0$, $a(S^1 \times S^2) = \frac{1}{2}$ and $0 < a(M) < \frac{1}{2}$ for every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$.
(2) The complete arith-genus $a(M)$ is a faithfully-complete rational invariant and we can reconstruct from the value of $a(M)$ the lattice point $\sigma_\alpha(M) \in \mathbb{X}$, the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself.

Let $a_{n,i} = a(M_{n,i})$. We have the following table on the complete arith-genera $a_{n,i}$ with $n \leqq 7$ by computing the data of Table 4.2.

Table 6.2.

$a_{0,1} = 0$

$a_{1,1} = \frac{1}{2} = 0.5$

$a_{3,1} = \frac{21}{4^3} = 0.328125$

$a_{4,1} = \frac{156}{5^4} = 0.2496$
$a_{4,2} = \frac{234}{5^4} = 0.3744$

$a_{5,1} = \frac{1555}{6^5} = 0.199974279...$
$a_{5,2} = \frac{2441}{6^5} = 0.313914609...$

$a_{6,1} = \frac{19608}{7^6} = 0.166665250...$
$a_{6,2} = \frac{31956}{7^6} = 0.271621518...$
$a_{6,3} = \frac{31842}{7^6} = 0.270652534...$
$a_{6,4} = \frac{36464}{7^6} = 0.309938886...$
$a_{6,5} = \frac{30960}{7^6} = 0.263155657...$
$a_{6,6} = \frac{38163}{7^6} = 0.324380147...$

$a_{6,7} = \frac{31863}{7^6} = 0.270831031...$
$a_{6,8} = \frac{46684}{7^6} = 0.396807452...$

$a_{7,1} = \frac{299593}{8^7} = 0.142857074...$
$a_{7,2} = \frac{500297}{8^7} = 0.238560199...$
$a_{7,3} = \frac{499127}{8^7} = 0.238002300...$
$a_{7,4} = \frac{488375}{8^7} = 0.232875347...$
$a_{7,5} = \frac{586679}{8^7} = 0.279750347...$
$a_{7,6} = \frac{585335}{8^7} = 0.279109478...$
$a_{7,7} = \frac{499319}{8^7} = 0.238093853...$
$a_{7,8} = \frac{733559}{8^7} = 0.349788188...$
$a_{7,9} = \frac{733577}{8^7} = 0.349796772...$
$a_{7,10} = \frac{732209}{8^7} = 0.349144458...$
$a_{7,11} = \frac{714289}{8^7} = 0.340599536...$

For any complex numbers u and z, we define the formal power series

$$\begin{aligned}\mu(u,z) &= \sum_{0\leqq n(\neq 2)<+\infty} \sum_{i=1}^{m_n} \frac{a_{n,i}}{n!(i-1)!} u^{i-1} z^n \\ &= \frac{1}{2}z + \frac{7}{2\cdot 4^3}z^3 + \frac{13}{2\cdot 5^4}z^4 + \frac{39}{4\cdot 5^4}uz^4 + \frac{311}{24\cdot 6^5}z^5 + \frac{2441}{120\cdot 6^5}uz^5 \\ &\quad + \frac{817}{30\cdot 7^6}z^6 + \frac{2663}{60\cdot 7^6}uz^6 + \frac{1769}{80\cdot 7^6}u^2z^6 + \frac{2279}{270\cdot 7^6}u^3z^6 + \frac{43}{24\cdot 7^6}u^4z^6 \\ &\quad + \frac{12721}{28800\cdot 7^6}u^5z^6 + \frac{10621}{172800\cdot 7^6}u^6z^6 + \frac{11671}{907200\cdot 7^6}u^7z^6 + \cdots .\end{aligned}$$

Then we have the following theorem.

Theorem 6.3. (1) For any complex numbers u and z, the formal power series $\mu(u,z)$ is absolutely convergent, so that $\mu(u,z)$ is a holomorphic function with $\mathbb{C}^2$ the absolute convergence domain.
(2) We have the identity

$$\frac{\partial^{i-1+n}}{\partial u^{i-1}\partial z^n}\mu(0,0) = a_{n,i}$$

for every $n \neq 2$ and $i \leqq m_n$, so that the holomorphic function $\mu(u,z)$ contains all the complete arith-genera $a_{n,i}$ ($n = 0,1,3,\ldots, i = 1,2,\ldots,m_n$) classifying all the 3-manifolds of $\mathbb{M}$.

Proof. It suffices to check that $\mu(u,z)$ is absolutely convergent for every $(u,z) \in \mathbb{C}^2$. In fact, we have

$$\sum_{0\leqq n(\neq 2)<+\infty} \sum_{i=1}^{m_n} \frac{a_{n,i}}{n!(i-1)!}|u^{i-1}|\cdot|z^n| \leqq \frac{1}{2}\sum_{n,i=0}^{+\infty} \frac{|u|^i}{i!}\frac{|z|^n}{n!} = \frac{1}{2}e^{|u|+|z|}. \quad \square$$

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