ON THE COHOMOLOGY OF TORUS MANIFOLDS

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Abstract. A *torus manifold* is an even-dimensional manifold acted on by a half-dimensional torus with non-empty fixed point set and some additional orientation data. It may be considered as a far-going generalisation of *toric manifolds* from algebraic geometry. The orbit space of a torus manifold has a reach combinatorial structure, e.g., it is a *manifold with corners* provided that the action is *locally standard*. Here we investigate relationships between the cohomological properties of torus manifolds and the combinatorics of their orbit quotients. We show that the cohomology ring of a torus manifold is generated by two-dimensional classes if and only if the quotient is a *homology polytope*. In this case one retrieves the familiar picture from toric geometry: the equivariant cohomology is the *face ring* of the nerve simplicial complex and the ordinary cohomology is obtained by factoring out certain linear forms. In a more general situation, we show that the odd-degree cohomology of a torus manifold vanishes if and only if the orbit space is *face-acyclic*. Although the cohomology is no longer generated in degree two under these circumstances, it is still possible to identify the equivariant cohomology with the face ring of an appropriate *simplicial poset*.

1. Introduction

Since the 1970s, algebraic geometers have studied equivariant algebraic compactifications of the *algebraic torus* $(\mathbb{C}^*)^n$, nowadays known as *complete toric varieties*. The study quickly grew in a separate branch of algebraic geometry, "toric geometry", incorporated many topological and convex-geometrical ideas and constructions, and produced a spectacular array of applications. A toric variety is a (normal) algebraic variety on which an algebraic torus acts with a dense orbit. The variety and the action are fully determined by a combinatorial object called *fan* [7].

With the appearance of pioneering work [6] of Davis and Januszkiewicz in the beginning of 1990s, the ideas of toric geometry have started penetrating into topology. The orbit space of a non-singular, complete (that is, compact), projective toric variety with respect to the action of the compact torus $T^n \subset (\mathbb{C}^n)^*$ can be identified with the simple polytope "dual" to the corresponding fan. Moreover, the action of the compact torus on a non-singular toric variety is "locally standard", that is, locally modelled by the diagonal action on $\mathbb{C}^n$. Davis and Januszkiewicz took these two characteristic properties as a starting point for their topological generalisation of toric varieties, *quasitoric manifolds*. A quasitoric manifold is a compact manifold M^{2n} with a locally standard action of T^n whose orbit space is (combinatorially) a simple polytope. (Davis and

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Januszkiewicz used the term "toric manifold", but by the time their work appeared the latter had already been used in algebraic geometry as a synonym of "non-singular toric variety".) According to one of the main results of [6], the cohomology ring of a quasitoric manifold M has the same structure as that of non-singular complete toric variety, and is isomorphic to the quotient of the Stanley–Reisner face ring of the orbit space by certain linear forms. In particular, the cohomology of M is generated by degree-two elements. For some further topological results related to quasitoric manifolds see [4] and [3, Ch.5].

At the same time, the convex-geometrical notion of polytope, while playing a very important role in geometrical considerations related to toric geometry, appears to be less relevant to the topological study of torus actions. The orbit quotient $Q = M/T$ of a non-singular compact toric variety M has a combinatorial structure of *manifold with corners* (this means that Q locally looks like the positive cone $\mathbb{R}^n_+$) in which all the faces, including Q itself, and all their intersections are acyclic. We call such a manifold with corners a *homology polytope*. It is a genuine polytope provided that the toric variety is projective, but in general may fail to be so. This implies, in particular, that the class of quasitoric manifolds does not cover all non-singular compact toric varieties (see [3, §5.2] for more discussion on the relationships between toric varieties and quasitoric manifolds). And one should expect that all the topological properties of quasitoric manifolds would still hold if one weakens the condition on the torus action by only requiring the orbit space to be a homology polytope. This is justified by certain results of the present paper (see Theorem 7.3 below).

An alternative far-going topological generalisation of complete non-singular toric varieties was introduced in [13] and [11] under the name of *torus manifolds* (or *unitary toric manifolds* in the earlier terminology). A torus manifold is an even-dimensional manifold M acted on by a half-dimensional torus T with non-empty fixed point set and some additional orientation data. Particular examples of torus manifolds include non-singular complete toric varieties (otherwise known as toric manifolds) and quasitoric manifolds of Davis and Januszkiewicz. On the other hand, the conditions on the action are significantly weakened in comparison to quasitoric manifolds. Surprisingly enough, torus manifolds admit a combinatorial treatment similar to toric varieties. It relies on the notions of *multi-fans* and *multi-polytopes*, developed in [11] as an alternative to fans associated with toric varieties.

The notion of torus manifold appears to be an appropriate template for investigating relationships between the topology of torus action and the combinatorics of orbit quotient, which is being the main theme of the current paper. Our first main result measures the extent of the analogy between the cohomological structure of non-singular complete toric varieties and torus manifolds. We show (Theorem 7.3) that the cohomology of a torus manifold M is generated by its degree-two part if and only if M is locally standard and the orbit space Q is a homology polytope. In this case the cohomology ring itself has a structure familiar from toric geometry: it is isomorphic to the Stanley–Reisner face ring of Q modulo certain linear forms. In particular, the cohomology is generated in degree two.

Next we study a more general class of torus manifolds with vanishing odd-degree cohomology. Under these circumstances the equivariant cohomology of M is a free finitely generated module over the equivariant cohomology of point, $H^*_T(pt) = \mathbb{Z}[t_1, \dots , t_n]$. This condition is known to algebraists as *Cohen–Macaulayness* and is equivalent to M being *equivariantly formal* in the terminology of [9]. The orbit space of a torus manifold with vanishing odd-degree cohomology needs not to be a homology polytope, as a simple example of torus acting on an even-dimensional sphere shows (see Example 2.4 below). We introduce a weaker notion of *face-acyclic* manifold with corners Q, in which all the faces are still acyclic, but their intersections may fail to be connected, and show (Theorem 8.3) that the odd-degree cohomology of M vanishes if and only if M is locally standard and the orbit space Q is face-acyclic. We also show that the equivariant cohomology is isomorphic to the face ring of the simplicial poset of faces of Q (note that the latter face ring is not generated by its degree-two elements in general), and identify the ordinary cohomology accordingly (Theorem 6.7 and Corollary 6.8).

At the end we establish Stanley's conjecture on the characterisation of h-vectors of Gorenstein* simplicial posets in the particular case of face posets of orbit quotients for torus manifolds (Theorem 9.2). Unlike the case of Gorenstein* simplicial complexes (which can be considered as an "algebraic approximation" to triangulations of spheres), the conditions for an integer vector to be an h-vector of a Gorenstein* simplicial poset are relatively weak. There are *Dehn–Sommerville equations* saying that the h-vector is symmetric, and its entries are non-negative. Apart from this, there are no conditions in odd dimensions. In even dimensions there is one more troublesome condition; the middle-dimensional entry of the h-vector must be even if at least one other entry is zero. It is not hard to check that these conditions are sufficient, by providing the corresponding examples of simplicial posets. Here we also show that those simplicial posets can be realised as face posets of orbit quotients for torus manifolds with vanishing odd-degree cohomology (so the corresponding h-vectors of posets are the even Betti vectors of torus manifolds). Stanley's conjecture [15] was that those conditions are also necessary. Although we were unable to prove this in full generality, we establish the conditions for h-vectors of posets associated to torus manifolds with vanishing odd-degree cohomology. This is done through the calculation of Stiefel–Whitney classes of torus manifolds.

We note that the characterisation of h-vectors for Gorenstein* simplicial complexes, as well as for sphere triangulations, remains wide open.

The paper is organised as follows. In Section 2 we introduce main topological concepts, give few examples, and establish some basic facts about torus manifolds. We also prove three pivotal statements (Lemmas 2.6–2.8) describing different properties of fixed point sets. In Section 3 we discuss locally standard torus actions. The main result here is Theorem 3.3 showing that a torus manifold M is locally standard provided that $H^{odd}(M) = 0$. We also introduce a canonical model for a torus manifold with given orbit space Q and distribution of circle subgroups fixing characteristic submanifolds. Then we show that a torus manifold is equivariantly diffeomorphic to its canonical model provided that $H^2(Q) = 0$. This extends the corresponding result for quasitoric manifolds due to Davis and Januszkiewicz. In Section 4 we develop the necessary apparatus of "combinatorial commutative algebra". Here we introduce face rings of manifolds with corners and simplicial posets, and list their main algebraic properties. We tried not to overload the notation with the poset terminology, but a reader familiar with posets would recognise the notions of (semi)lattice, meet, join, etc. In Section 5 we turn to the equivariant cohomology of torus manifolds. We introduce certain key concepts and construct a map from the face ring of orbit quotient to the equivariant cohomology of torus manifold, which is later shown to be an isomorphism under certain conditions. Sections 6–8 contain the proofs of the main results quoted above. In Section 9 we prove the above mentioned particular case of Stanley's conjecture on Gorenstein* simplicial posets.

2. Torus manifolds

The notation below is that of [11] and [13], with some additional specifications. For a topological space X with a topological action of a topological group G, its *equivariant cohomology* with $\mathbf{k}$ coefficient ($\mathbf{k}$ a ring) is defined as

$$H^*_G(X;\mathbf{k}) := H^*(EG\times_G X;\mathbf{k})$$

where EG is the total space of a universal principal G-bundle (on which G acts freely) and $EG\times_G X$ is the orbit space of $EG\times X$ by the diagonal action. The space $EG\times_G X$ is sometimes called the *Borel construction* on the G-space X. Borel construction can be performed for a G-vector bundle. For instance, if E is an oriented G-vector bundle over a G-space X, then the Borel construction on E produces an oriented vector bundle over $EG\times_G X$ and its Euler class is called the *equivariant Euler class* of E and denoted by $e^G(E)$. Note that $e^G(E)$ lies in $H^*_G(X;\mathbb{Z})$. Below we take integer coefficients, unless another coefficient ring is specified in the notation.

2.1. **Torus manifolds.** Let M be a $2n$-dimensional closed connected orientable smooth manifold with an effective smooth action of an n-dimensional torus $T=(S^1)^n$ such that $M^T\neq\varnothing$. Since $\dim M = 2\dim T$ and M is compact, the fixed point set M^T is a finite set of isolated points.

The map ρ collapsing M to a point induces a homomorphism $\rho^*\colon H^*_T(pt) = H^*(BT)\to H^*_T(M)$ defining a canonical $H^*(BT)$-module structure on $H^*_T(M)$. As is well known, $H^*(BT)$ is a polynomial ring in n variables of degree two, in particular $H^{odd}(BT)=0$. Since the fixed point set M^T is non-empty, the map ρ^* is injective. However, $H^*_T(M)$ is not necessarily free as an $H^*(BT)$-module.

Lemma 2.1. *$H^*_T(M)$ is free as an $H^*(BT)$-module (in fact, $H^*_T(M)$ is isomorphic to $H^*(BT)\otimes H^*(M)$ as an $H^*(BT)$-module) if and only if $H^{odd}(M)=0$.*

Proof. Suppose $H^{odd}(M)=0$. Then the Serre spectral sequence of the fibration $\rho\colon ET\times_T M\to BT$ collapses and $H^*(M)$ has no torsion, so $H^*_T(M)$ is isomorphic to $H^*(BT)\otimes H^*(M)$ and free as an $H^*(BT)$-module. This proves the "if" part.

To prove the "only if" part, we use the Eilenberg–Moore spectral sequence of the bundle $ET \times_T M \to BT$ with fibre M. It converges to $H^*(M)$ and has

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(BT)}\big(H_T^*(M), \mathbb{Z}\big).$$

Since $H_T^*(M)$ is free as an $H^*(BT)$-module, we have

$$\begin{aligned}\mathrm{Tor}^{*,*}_{H^*(BT)}\big(H_T^*(M), \mathbb{Z}\big) &= \mathrm{Tor}^{0,*}_{H^*(BT)}\big(H_T^*(M), \mathbb{Z}\big)\\ &= H_T^*(M) \otimes_{H^*(BT)} \mathbb{Z}\\ &= H_T^*(M)\big/(\rho^*(H^{>0}(BT))).\end{aligned}$$

Therefore, $E_2^{0,*} = H_T^*(M)\big/(\rho^*(H^{>0}(BT)))$ and $E_2^{-p,*} = 0$ for $p > 0$. It follows that the Eilenberg–Moore spectral sequence collapses at the E_2 term and

$$H^*(M) = H_T^*(M)\big/(\rho^*(H^{>0}(BT))). \tag{2.2}$$

On the other hand, it follows from the localization theorem (see [12]) that the kernel of the restriction map

$$H_T^*(M) \to H_T^*(M^T) = H^*(BT) \otimes H^*(M^T)$$

is the $H^*(BT)$-torsion subgroup and hence the restriction map is injective in our case. Therefore $H_T^{odd}(M) = 0$ because M^T is a finite set of isolated points. This fact together with (2.2) proves that $H^{odd}(M) = 0$. □

A closed, connected, codimension-two submanifold of M is called *characteristic* if it is a connected component of the set fixed pointwise by a certain circle subgroup of T and contains at least one T-fixed point. Since M is compact, there are only finitely many characteristic submanifolds. We denote them by M_i ($i = 1, \dots, m$). For $2 \leqslant k \leqslant n$, k different characteristic submanifolds intersect transversally (if their intersection is non-empty) and the intersection is a disjoint union of finite number of codimension-$2k$ submanifolds fixed pointwise by a codimension k subtorus. In particular, an intersection of n characteristic submanifolds consists of finitely many T-fixed points.

Each M_i is orientable. Following [4], we say that M is *omnioriented* if an orientation is specified for M and for every characteristic submanifold M_i. There are 2^{m+1} choices of omniorientations and following [11] we say that M is a *torus manifold* when it is omnioriented.

Here are two typical examples of torus manifolds.

Example 2.3. A complex projective space $\mathbb{C}P^n$ has a natural T-action defined in the homogeneous coordinates by

$$(t_1, \dots, t_n) \cdot (z_0 : z_1 : \dots : z_n) = (z_0 : t_1 z_1 : \dots : t_n z_n).$$

It has $(n+1)$ characteristic submanifolds $z_0 = 0, \dots, z_n = 0$ and $(n+1)$ fixed points $(1 : 0 : \dots : 0), \dots, (0 : \dots : 0 : 1)$. In this example the intersection of any set of characteristic submanifolds is connected.

Example 2.4. Let S^{2n} be the $2n$-sphere identified with the following subset in $\mathbb{C}^n \times \mathbb{R}$:

$$\big\{(z_1, \dots, z_n, y) \in \mathbb{C}^n \times \mathbb{R} \colon |z_1|^2 + \dots + |z_n|^2 + y^2 = 1\big\},$$

and define a T-action by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n, y) = (t_1 z_1, \dots, t_n z_n, y).$$

This has n characteristic submanifolds $z_1 = 0, \dots, z_n = 0$, and two fixed points $(0, \dots, 0, \pm 1)$. An intersection of k characteristic submanifolds is connected if $k \leqslant n-1$, but disconnected (in fact, consists of the two fixed points) if $k = n$.

If M is a torus manifold, then both M and M_i are oriented; so we have a Gysin homomorphism $H^*_T(M_i) \to H^{*+2}_T(M)$ in the equivariant cohomology. Denote by $\tau_i \in H^2_T(M)$ the image of the identity element in $H^0_T(M_i)$. We may think of τ_i as the Poincaré dual of M_i in the equivariant cohomology.

Proposition 2.5 (See section 1 of [13])**.** *Let M be a torus manifold.*

1. *For each $i = 1, \dots, m$, there is a unique element $a_i \in H_2(BT)$ such that*
$$\rho^*(t) = \sum_{i=1}^{m} \langle t, a_i \rangle \tau_i \quad \text{modulo } H^*(BT)\text{-torsions}$$
for any element $t \in H^2(BT)$.
2. *The circle subgroup fixing the characteristic submanifold M_i coincides with that determined by $a_i \in H_2(BT)$ via the identification $H_2(BT) = \mathrm{Hom}(S^1, T)$.*
3. *If n different characteristic submanifolds $M_{i_1}, \dots, M_{i_n}$ have a T-fixed point in their intersection, then the elements $a_{i_1}, \dots, a_{i_n}$ form a basis of $H_2(BT)$ over $\mathbb{Z}$.*

2.2. **Cohomology of fixed point sets.** In this paper we are mainly concerned with torus manifolds which have vanishing odd degree cohomology or more strongly have cohomology generated in degree two, see Examples 2.3, 2.4. The following two lemmas show that these cohomological properties are inherited by connected components of the fixed point set M^H for any subtorus $H \subset T$. These facts will enable us to use induction argument on dimension in later sections.

Lemma 2.6. *Let M be a torus manifold, H a subtorus of T and N a connected component of M^H. If $H^{odd}(M) = 0$, then $H^{odd}(N) = 0$ and hence $N^T \neq \varnothing$.*

Proof. The first statement is equivalent to $H^{odd}(M^H) = 0$. We note that if S^1 is a generic circle subgroup of H, then $M^{S^1} = M^H$. Let p be a prime and G be an order p subgroup of the generic circle subgroup S^1. The induced action of G on $H^*(M)$ is trivial because G is contained in the connected group S^1. Moreover $H^{odd}(M; \mathbb{Z}/p) = 0$ by assumption. It follows from [1, Theorem VII.2.2] that $H^{odd}(M^G; \mathbb{Z}/p) = 0$. Repeating the same argument for M^G with the induced action of S^1/G which is again a circle group, one concludes that $H^{odd}(M^G; \mathbb{Z}/p) = 0$ for any p-subgroup G of S^1. However, $M^G = M^{S^1} (= M^H)$ if the order of G is sufficiently large, so we have $H^{odd}(M^H; \mathbb{Z}/p) = 0$. Since p is an arbitrary prime, this implies that $H^{odd}(M^H) = 0$.

Finally, since $H^{odd}(N) = 0$, the Euler characteristic $\chi(N)$ of N is non-zero. As is well-known $\chi(N) = \chi(N^T)$, so N^T must be non-empty. □

Lemma 2.7. *Let M, H, N be as in Lemma 2.6. If $H^*(M)$ is generated by its degree-two part (as a ring), then the restriction map $H^*(M) \to H^*(N)$ is surjective; in particular, $H^*(N)$ is also generated by its degree-two part.*

Proof. Since $H^{odd}(M) = 0$, we have $H^{odd}(N) = 0$ by Lemma 2.6; so it suffices to prove that the restriction map $H^*(M; \mathbb{Z}/p) \to H^*(N; \mathbb{Z}/p)$ is surjective for any prime p.

The argument below is similar to that used in the proof of Theorem VII.3.1 in [1, p. 379]. As in the proof of Lemma 2.6, let S^1 be a generic circle subgroup of H (so that $M^{S^1} = M^H$) and let G be the subgroup of S^1 of prime order p. Then the restriction map $H^k_G(M; \mathbb{Z}/p) \to H^k_G(M^G; \mathbb{Z}/p)$ is an isomorphism for sufficiently large k (see [1, Theorem VII.1.5]). Hence, for any connected component N' of M^G the restriction $r\colon H^k_G(M; \mathbb{Z}/p) \to H^k_G(N'; \mathbb{Z}/p)$ is surjective if k is sufficiently large. Now consider the commutative diagram

$$\begin{array}{ccc} H^*_G(M; \mathbb{Z}/p) & \xrightarrow{r} & H^*_G(N'; \mathbb{Z}/p) \cong H^*(BG; \mathbb{Z}/p) \otimes H^*(N'; \mathbb{Z}/p) \\ \downarrow & & \downarrow \\ H^*(M; \mathbb{Z}/p) & \xrightarrow{s} & H^*(N'; \mathbb{Z}/p) \end{array}.$$

By the assumption, $H^*(M; \mathbb{Z}/p)$ is generated by elements in $H^2(M; \mathbb{Z}/p)$, say $v_1, \dots, v_d$. Since $H^{odd}(M; \mathbb{Z}/p) = H^{odd}(M^G; \mathbb{Z}/p) = 0$ and $\chi(M) = \chi(M^T) = \chi(M^G)$, we have $\sum \operatorname{rank} H^i(M; \mathbb{Z}/p) = \sum \operatorname{rank} H^i(M^G; \mathbb{Z}/p)$. Therefore, the vertical map $H^*_G(M; \mathbb{Z}/p) \to H^*(M; \mathbb{Z}/p)$ in the above diagram is surjective (see [1, Theorem VII.1.6]). Let $\xi_j \in H^*_G(M; \mathbb{Z}/p)$ be a lift of v_j, and $w_j := s(v_j)$. Let t be a non-zero element in $H^2(BG; \mathbb{Z}/p)$. Any power of t is non-zero as is well known. Since the above diagram is commutative and $H^1(N'; \mathbb{Z}/p) = 0$ as shown in the proof of Lemma 2.6, we have $r(\xi_j) = \alpha_j t + w_j$ for some $\alpha_j \in \mathbb{Z}/p$. Now let $a \in H^*(N'; \mathbb{Z}/p)$ be an arbitrary element. Then there exist ℓ and a polynomial $P(\xi_1, \dots, \xi_d)$ such that

$$r\big(P(\xi_1, \dots, \xi_d)\big) = t^\ell a.$$

On the other hand,

$$r\big(P(\xi_1, \dots, \xi_d)\big) = P(\alpha_1 t + w_1, \dots, \alpha_d t + w_d) = \sum_{k \geqslant 0} t^k Q_k(w_1, \dots, w_d)$$

for some polynomials Q_k. Therefore, $a = Q_\ell(w_1, \dots, w_d)$, the restriction map $H^*(M; \mathbb{Z}/p) \to H^*(N'; \mathbb{Z}/p)$ is surjective and $H^*(N'; \mathbb{Z}/p)$ is generated by the degree-two elements $w_1, \dots, w_d$.

Repeating the same argument for N' with the induced action of S^1/G which is again a circle group, one concludes that the restriction map $H^*(M; \mathbb{Z}/p) \to H^*(N'; \mathbb{Z}/p)$ is surjective for any connected component N' of M^G with G any p-subgroup of S^1. However, if the order of G is sufficiently large, then $M^G = M^{S^1} (= M^H)$ and hence $N' = N$, so it follows that the restriction map $H^*(M; \mathbb{Z}/p) \to H^*(N; \mathbb{Z}/p)$ is surjective for any connected component N of M^H. Since the prime p is arbitrary, this implies the lemma. ∎

2.3. **Intersection of characteristic submanifolds.** A multiple intersection of characteristic submanifolds fails to be connected in Example 2.4, but all multiple intersections of characteristic submanifolds are connected in Example 2.3. In fact, we have the following statement.

Lemma 2.8. *Suppose that $H^*(M)$ is generated in degree two. Then all non-empty multiple intersections of the characteristic submanifolds are connected and have cohomology generated in degree two.*

Proof. Since every characteristic submanifold M_i is a connected component of the fixed point set of a circle subgroup of T, the cohomology $H^*(M_i)$ is generated by the degree-two part and the restriction map $H^*(M) \to H^*(M_i)$ is onto by Lemma 2.7. Then the restriction map $H^*_T(M) \to H^*_T(M_i)$ in equivariant cohomology is also onto.

Now we prove the connectedness of multiple intersections. Suppose that the intersection of k $(1 < k \leqslant n)$ different characteristic submanifolds, $M_{i_1} \cap \dots \cap M_{i_k}$, is non-empty. Since $M_{i_1} \cap \dots \cap M_{i_k}$ is fixed by a subtorus, every connected component of $M_{i_1} \cap \dots \cap M_{i_k}$ has a T-fixed point by Lemma 2.6. Let N be a connected component of $M_{i_1} \cap \dots \cap M_{i_k}$. For every $i \in \{i_1, \dots, i_k\}$ let us consider the embeddings $\varphi_i \colon N \to M_i$ and $\psi_i \colon M_i \to M$, and look at the corresponding Gysin homomorphisms in the equivariant cohomology:

$$H^0_T(N) \xrightarrow{\varphi_{i!}} H^{2k-2}_T(M_i) \xrightarrow{\psi_{i!}} H^{2k}_T(M).$$

Since the restriction $\psi_i^* \colon H^*_T(M) \to H^*_T(M_i)$ is surjective, we have $\varphi_{i!}(1) = \psi_i^*(u)$ for some $u \in H^{2k-2}_T(M)$. Now we calculate

$$(\psi_i \circ \varphi_i)_!(1) = \psi_{i!}(\varphi_{i!}(1)) = \psi_{i!}\big(\psi_i^*(u)\big) = \psi_{i!}(1)u = \tau_i u.$$

Hence, $(\psi_i \circ \varphi_i)_!(1)$ is divisible by τ_i for every $i \in \{i_1, \dots, i_k\}$. By Proposition 3.4 of [13], the degree-$2k$ part of $H^*_T(M)$ is additively generated by monomials $\tau_{j_1}^{k_1} \dots \tau_{j_p}^{k_p}$ with $k_1 + \dots + k_p = k$ and $\{j_1, \dots, j_p\}$ such that $M_{j_1} \cap \dots \cap M_{j_p} \neq \varnothing$. It follows that $(\psi_i \circ \varphi_i)_!(1)$ is a non-zero integral multiple of $\tau_{i_1} \dots \tau_{i_k} \in H^{2k}_T(M)$. By the definition of Gysin map, $(\psi_i \circ \varphi_i)_!(1)$ maps to zero under the restriction map $H^*_T(M) \to H^*_T(x)$ for every point $x \in (M \backslash N)^T$. On the other hand, the image of $\tau_{i_1} \dots \tau_{i_k}$ under the restriction map $H^*_T(M) \to H^*_T(x)$ is non-zero for every T-fixed point $x \in M_{i_1} \cap \dots \cap M_{i_k}$. Thus, N is the only connected component of the latter intersection. The second statement in the lemma follows from Lemma 2.7. □

3. Locally standard torus manifolds and orbit spaces

Let $Q := M/T$ denote the orbit space of M and $\pi \colon M \to Q$ the quotient projection. Define the *facets* of Q to be the orbit spaces of characteristic submanifolds: $Q_i := \pi(M_i)$, $i = 1, \dots, m$. Every facet is a closed connected subset in Q of "codimension one". We refer to a non-empty intersection of k facets as a codimension-k *preface*, $k = 1, \dots, n$. Hence, a preface is the orbit space of some non-empty intersection $M_{i_1} \cap \dots \cap M_{i_k}$ of characteristic submanifolds. If $H^*(M)$ is generated in degree two, then all prefaces are connected by Lemma 2.8, but in general, prefaces of codimension > 1 may fail to be connected (see Example 2.4). We refer to connected components of prefaces as *faces*. We also regard Q itself as a codimension-zero face; other faces are called *proper faces*. A space X is *acyclic* if $\widetilde{H}_i(X) = 0$ for all i. We say that Q is *face-acyclic* if all of its faces (including Q itself) are acyclic. We call Q a *homology polytope* if all its prefaces are acyclic (in particular, connected). Note that $Q = M/T$ is a homology polytope if and only if it is face-acyclic and all non-empty multiple intersections of characteristic submanifolds M_i are connected.

3.1. **Locally standardness.** We say that a torus manifold M is *locally standard* if every point in M has an invariant neighbourhood U weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ (invariant under the standard T^n-action on $\mathbb{C}^n$). The latter means that there is an automorphism $\psi\colon T \to T$ and a diffeomorphism $f\colon U \to W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T,\ y \in U$.

Any point in the orbit space Q of a locally standard torus manifold M has a neighbourhood diffeomorphic to an open subset in the positive cone

$$\mathbb{R}^n_+ = \{(y_1, \dots, y_n) \in \mathbb{R}^n \colon y_i \geqslant 0,\ i = 1, \dots, n\}.$$

Moreover, this local diffeomorphism preserves the face structures in Q and $\mathbb{R}^n_+$ (that is, a point from a codimension-k face of Q is mapped to a point with at least k zero coordinates). By the definition, this identifies Q as a *manifold with corners*, see e.g. [5, §6]. In particular, Q is a manifold with boundary $\partial Q = \cup_i Q_i$. Let K denote the *nerve* of the covering of ∂Q by the facets. Thus, K is an $(n-1)$-dimensional simplicial complex on m vertices. The $(k-1)$-dimensional simplices of K are in one-to-one correspondence with the codimension-k prefaces of Q.

Remark. A simple convex polytope is an example of a manifold with corners and it is a homology polytope. A *quasitoric manifold* [6], [3] can be defined as a locally standard torus manifold (forgetting the omniorientation) whose orbit space is a *simple convex polytope* with the standard face structure. The nerve is the boundary complex of the dual simplicial polytope.

Example 3.1. The torus manifold $\mathbb{C}P^n$ with the T-action in Example 2.3 is locally standard and the map

$$(z_0 : z_1 : \dots : z_n) \to \frac{1}{\sum_{i=0}^n |z_i|^2}(|z_1|^2, \dots, |z_n|^2)$$

induces a face preserving homeomorphism from the orbit space $\mathbb{C}P^n/T$ to a standard n-simplex which is a simple polytope, in particular, a homology polytope.

Example 3.2. The torus manifold S^{2n} with the T-action in Example 2.4 is also locally standard and the map

$$(z_1, \dots, z_n, y) \to (|z_1|, \dots, |z_n|, y)$$

induces a face preserving homeomorphism from the orbit space S^{2n}/T to the space

$$\{(x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} \colon x_1^2 + \dots + x_n^2 + y^2 = 1,\ x_1 \geqslant 0, \dots, x_n \geqslant 0\}.$$

This space is not a simple polytope (even not a homology polytope) but a manifold with corners and face-acyclic.

Here is a sufficient condition for a torus manifold M to be locally standard.

Theorem 3.3. *A torus manifold M is locally standard if $H^{odd}(M) = 0$.*

Proof. **Step 1.** We show that no point in M has a non-trivial finite isotropy group. Suppose that there is a point $x \in M$ whose isotropy group T_x is a non-trivial finite group. Then, T_x contains a non-trivial cyclic subgroup G of prime order, say p, and let N be the connected component of M^G containing the point x. Since N contains x and T_x is finite, the principal isotropy group of N is finite. On the other hand, the proof of Lemma 2.6 shows that $H^{odd}(N;\mathbb{Z}/p) = 0$, in particular, the Euler characteristic of N is non-zero. Therefore N has a T-fixed point, say y. The tangential T-representation $\mathcal{T}_yM$ at y is faithful, $\dim M = 2\dim T$ and $\mathcal{T}_yN$ is a proper T-subrepresentation of $\mathcal{T}_yM$. It follows that there is a subtorus T' (of positive dimension) which fixes $\mathcal{T}_yN$ and does not fix the complement of $\mathcal{T}_yN$ in $\mathcal{T}_yM$. Clearly, T' is the principal isotropy group of N and this contradicts the observation above that the principal isotropy group of N is finite.

Step 2. If the isotropy group T_x is trivial, then it is obvious that M is locally standard around the point x. Suppose that the isotropy group T_x is non-trivial. Then $\dim T_x > 0$ by Step 1. Let H be the identity component of T_x and let N be the connected component of M^H containing x. By Lemma 2.6, N has a T-fixed point, say y. Looking at the tangential representation at y, one sees that the induced action of T/H on N is effective. By Step 1, no point of N has a non-trivial finite isotropy group for the induced action of T/H. This means that $T_x = H$ and since x and y are both in the same connected component N fixed pointwise by T_x, the T_x-representation in $\mathcal{T}_xM$ agrees with the restriction of the tangential T-representation $\mathcal{T}_yM$ to T_x. This implies that M is locally standard around x. □

3.2. **Canonical model.** We assume that a torus manifold M is locally standard. Then the orbit space Q is a manifold with corners. The facets of Q are the quotient images Q_i of characteristic submanifolds M_i $(i = 1, \dots, m)$. Let

$$\Lambda\colon \{1,\dots,m\} \to H_2(BT) = \operatorname{Hom}(S^1,T) \cong \mathbb{Z}^n \tag{3.4}$$

be a map sending i to a_i in Proposition 2.5. The circle subgroup determined by a_i, that is $a_i(S^1)$, is the one which fixes M_i. Because of Proposition 2.5, the map Λ satisfies the following *non-singular condition*:

> If $Q_{i_1} \cap \dots \cap Q_{i_k}$ is non-empty, then $\Lambda(i_1), \dots, \Lambda(i_k)$ span
> a k-dimensional unimodular subspace of $\operatorname{Hom}(S^1,T) \cong \mathbb{Z}^n$.

Given a point $x \in Q$, the smallest face which contains x is an intersection of some facets $Q_{i_1} \cap \dots \cap Q_{i_k}$, and we define $T(x)$ to be the subtorus of T generated by circle subgroups $\Lambda(i_1)(S^1), \dots, \Lambda(i_k)(S^1)$. We introduce the identification space

$$M_Q(\Lambda) := T \times Q/\sim, \tag{3.5}$$

where $(t,x) \sim (t',x')$ if and only if $x = x'$ and $t^{-1}t' \in T(x)$. The space $M_Q(\Lambda)$ admits a natural action of T and is a closed manifold because the map Λ satisfies the non-singular condition above and Q is a manifold with corners. The following is a straightforward generalisation of a Davis–Januszkiewicz result [6, Prop. 1.8].

Lemma 3.6. *Let M be a locally standard torus manifold with orbit space Q and the map Λ in (3.4). If $H^2(Q) = 0$, then there is an equivariant homeomorphism*

$$M_Q(\Lambda) \to M$$

covering the identity on Q.

Remark. Like in the Davis–Januszkiewicz case, it follows that a torus manifold whose orbit quotient Q satisfies $H^2(Q) = 0$ is determined by Q and Λ.

Proof. The idea is to construct a continuous map $f\colon T \times Q \to M$ taking $T \times q$ onto $\pi^{-1}(q)$ for each point $q \in Q$. This is done by subsequent "blowing up the singular strata". The condition on the second cohomology group guarantees that the resulting principal T-bundle over Q is trivial. Then the map f descends to the required equivariant homeomorphism. See [6] for details. □

4. Face rings of manifolds with corners and simplicial posets

To study the (equivariant) cohomology rings of torus manifolds we need an algebraic digression. Here we review a notion of face ring generalising the classical Stanley–Reisner face ring to combinatorial structures more general than simplicial complexes. We consider two cases, which are in a sense dual to each other: "nice" manifolds with corners and simplicial posets. The latter one is more general, however for applications to torus manifolds we just need the former one. The face ring of a manifold with corners is also a little easier to visualise, so we start with considering this case.

The relationship between nice manifolds with corners and simplicial posets is similar to that between simple polytopes and simplicial complexes. Face rings of simplicial posets were introduced and studied in [15]. Most of the statements in this section follow from the general theory of ASL's (*algebras with straightening law*) and *Hodge algebras* as explained in [15] and [2, Ch. 7], however our treatment is independent and geometrical.

4.1. Nice manifolds with corners.

First assume that Q is a homology polytope (or even a simple convex polytope) with m facets $Q_1, \dots, Q_m$. Let $\mathbf{k}$ be a ground commutative ring with unit. Then the Stanley–Reisner face ring of its nerve K can be identified with the ring

$$\mathbf{k}[Q] = \mathbf{k}\big[v_{Q_1}, \dots, v_{Q_m}\big]\big/\big(v_{Q_{i_1}} \cdots v_{Q_{i_k}} = 0 \quad \text{if} \quad Q_{i_1} \cap \cdots \cap Q_{i_k} = \varnothing\big).$$

We refer to $\mathbf{k}[Q]$ as the *face ring of Q*.

For arbitrary pair of faces G, H of Q the intersection $G \cap H$ is a unique maximal face contained in both G and H. On the other hand, there is a unique minimal face that contains both G and H, which we denote $G \vee H$. Let $\mathbf{k}[v_F\colon F \text{ a face}]$ be the graded polynomial ring with one $2k$-dimensional generator v_F for every proper codimension-k face F. We also identify v_Q with the unit and $v_\varnothing$ with zero. The following proposition gives another presentation of $\mathbf{k}[Q]$, by extending both the set of generators and relations, and will be used for a subsequent generalisation of $\mathbf{k}[Q]$ to arbitrary manifolds with corners.

Proposition 4.1. *There is a canonical isomorphism of rings*

$$\mathbf{k}[v_F\colon F \ \text{a face}]/\mathcal{I}_Q \cong \mathbf{k}[Q],$$

where $\mathcal{I}_Q$ is the ideal generated by all

$$v_G v_H - v_{G\vee H} v_{G\cap H}.$$

Proof. The identification is established by the map sending v_F to $\prod_{Q_i\supset F} v_{Q_i}$. □

Now let Q be an arbitrary connected manifold with corners. We also assume that Q is *nice*, that is, every codimension-k face is contained in exactly k facets. Note that the orbit space of a locally standard torus manifold is always nice. In a nice manifold with corners, all faces containing a given face form a Boolean lattice (like in the case of $\mathbb{R}^n_+$).

Remark. By the definition of manifold with corners, every codimension-k face is contained in at most k facets. A 2-disc with one 0-face and one 1-face on the boundary gives an example of manifold with corners which is not nice.

The intersection of two faces G and H in a manifold with corners may be disconnected, but every its connected component is a face of codimension $\operatorname{codim} G + \operatorname{codim} H$. We regard $G\cap H$ as the set of its connected components; so the notation $E \in G\cap H$ is used below for connected components E in the intersection. The following proposition shows that the face $G\vee H$ is still well-defined provided that $G\cap H \neq \varnothing$.

Proposition 4.2. *For every two faces G and H with non-empty intersection, there is a unique minimal face $G\vee H$ that contains both G and H.*

Proof. Take any $E \in G\cap H$. The statement follows from the fact that the poset of faces containing E is a Boolean lattice. □

Now we use the interpretation from Proposition 4.1 to introduce a more general version of $\mathbf{k}[Q]$.

Definition 4.3. The *face ring* of a nice manifold with corners Q is the quotient

$$\mathbf{k}[Q] := \mathbf{k}[v_F\colon F \ \text{a face}]/\mathcal{I}_Q,$$

where $\mathcal{I}_Q$ is the ideal generated by all

$$v_G v_H - v_{G\vee H}\cdot \sum_{E\in G\cap H} v_E.$$

In particular, if G and H are transversal, that is, $\operatorname{codim} G\cap H = \operatorname{codim} G + \operatorname{codim} H$, then $G\vee H = Q$, so we get the identity

$$v_G v_H = \sum_{E\in G\cap H} v_E \quad \text{in } \mathbf{k}[Q].$$

Remark. The nerve K of Q is a simplicial complex and the face ring $\mathbf{k}[Q]$ agrees with the classical Stanley–Reisner face ring $\mathbf{k}[K]$ of the simplicial complex K if all non-empty multiple intersections of facets in Q are connected, but otherwise they differ in general.

Below we give a sequence of statements describing algebraic properties of $\mathbf{k}[Q]$ and emphasising its analogy with the classical Stanley–Reisner face ring.

Lemma 4.4. *Every element $a \in \mathbf{k}[Q]$ can be written as*

$$a = \sum_{G_1 \supset \cdots \supset G_n} A\, v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}$$

where $A \in \mathbf{k}$ are some coefficients, the sum is taken over all chain of faces $G_1 \supset \cdots \supset G_n$ with $\operatorname{codim} G_i = i$, *and α_i are some non-negative integers.*

Proof. We may assume that $a = v_{H_1} v_{H_2} \cdots v_{H_k}$ (some H_i may coincide), and it is enough to show that it can be written as $\sum v_{G_1} \cdots v_{G_l}$ with $G_1 \supset \cdots \supset G_l$ for every summand (without making any assumptions on codimensions, but allowing some G_i to coincide). By induction we may assume that $H_2 \supset \cdots \supset H_k$. Now we apply the relation from Definition 4.3 and replace a by

$$v_{H_1 \vee H_2} \Big(\sum_{E \in H_1 \cap H_2} v_E \Big) v_{H_3} \cdots v_{H_k}.$$

Now the first two faces in every summand are ordered. Then we replace each $v_E v_{H_3}$ by $v_{E \vee H_3} (\sum_{G \in E \cap H_3} v_G)$. Since $H_1 \vee H_2 \supset E \vee H_3$, we get the first three faces in a linear order. Proceeding in this fashion we finally end up in a sum of monomials corresponding to ordered sets of faces. □

We refer to the presentation from Lemma 4.4 as the *chain decomposition* of an element $a \in \mathbf{k}[Q]$

For any vertex (0-face) $p \in Q$ we define the *restriction map* s_p by

$$s_p \colon \mathbf{k}[Q] \to \mathbf{k}[Q]/(v_F : F \not\ni p).$$

The next observation is straightforward.

Proposition 4.5. *The image $s_p(\mathbf{k}[Q])$ of the restriction map can be identified with the polynomial ring $\mathbf{k}[v_{Q_{i_1}}, \ldots, v_{Q_{i_n}}]$ of n degree-two generators, where $Q_{i_1}, \ldots, Q_{i_n}$ are the n different facets containing p.*

Lemma 4.6. *The sum $s = \oplus_p s_p$ of restriction maps over all vertices $p \in Q$ is a monomorphism from $\mathbf{k}[Q]$ to the sum of polynomial rings.*

Proof. Take a non-zero $a \in \mathbf{k}[Q]$ and write it as in Lemma 4.4. Fix a monomial $A\, v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}$ with non-zero coefficient A and consider the restriction s_p to the vertex $p = G_n$. We claim that $s_p(a) \neq 0$. Identify $s_p(\mathbf{k}[Q])$ with the polynomial ring $\mathbf{k}[t_1, \ldots, t_n]$ (so that $t_j := v_{Q_{i_j}}$ in the notation of Proposition 4.5). Then $s_p(v_{G_n}) = t_1 \cdots t_n$ and we may also assume that $s_p(v_{G_j}) = t_1 \cdots t_j,\ j = 1, \ldots, n$. Hence,

$$s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}\big) = t_1^{\alpha_1} (t_1 t_2)^{\alpha_2} \cdots (t_1 \cdots t_n)^{\alpha_n}.$$

It follows that $s_p(a)\ne 0$ unless some other monomial $v_{H_1}^{\beta_1}\cdots v_{H_n}^{\beta_n}$ hits the same monomial in $\mathbf{k}[t_1,\dots,t_n]$. Note that

$$s_p(v_{H_1}^{\beta_1}\cdots v_{H_n}^{\beta_n})=0 \qquad \text{unless } H_k\supset G_n \text{ for } \beta_k\ne 0.$$

Suppose

$$s_p\bigl(v_{G_1}^{\alpha_1}\cdots v_{G_n}^{\alpha_n}\bigr)=s_p(v_{H_1}^{\beta_1}\cdots v_{H_n}^{\beta_n}). \tag{4.7}$$

We want to prove that $v_{G_1}^{\alpha_1}\cdots v_{G_n}^{\alpha_n}=v_{H_1}^{\beta_1}\cdots v_{H_n}^{\beta_n}$, that is, $\alpha_i=\beta_i$ and $G_i=H_i$ if $\alpha_i\ne 0,\ \ i=1,\dots,n$. By induction, we may prove this for $i=j$ assuming that it is true for $i>j$. Then (4.7) turns to the identity

$$\begin{aligned} s_p\bigl(v_{G_1}^{\alpha_1}\cdots v_{G_j}^{\alpha_j}\bigr)&(t_1\dots t_{j+1})^{\alpha_{j+1}}\cdots(t_1\cdots t_n)^{\alpha_n}\\ &=s_p\bigl(v_{H_1}^{\beta_1}\cdots v_{H_j}^{\beta_j}\bigr)(t_1\dots t_{j+1})^{\alpha_{j+1}}\cdots(t_1\cdots t_n)^{\alpha_n},\end{aligned}$$

whence $s_p(v_{G_1}^{\alpha_1}\cdots v_{G_j}^{\alpha_j})=s_p(v_{H_1}^{\beta_1}\cdots v_{H_j}^{\beta_j})$. Suppose that β_l is the last non-zero exponent (so that $\beta_{l+1}=\cdots=\beta_j=0$). Then we also have $\alpha_{l+1}=\cdots=\alpha_j=0$, since otherwise $s_p(v_{G_1}^{\alpha_1}\cdots v_{G_j}^{\alpha_j})$ would be divisible by $t_1\dots t_{l+1}$, while $s_p(v_{H_1}^{\beta_1}\cdots v_{H_j}^{\beta_j})$ is not. We also have $\alpha_l=\beta_l$ and $G_l=H_l$ since α_l is the maximal power of $t_1\dots t_l$ that divides $s_p(v_{G_1}^{\alpha_1}\cdots v_{G_j}^{\alpha_j})$. By induction, we conclude that $v_{G_1}^{\alpha_1}\cdots v_{G_n}^{\alpha_n}=v_{H_1}^{\beta_1}\cdots v_{H_n}^{\beta_n}$, whence $s_p(a)\ne 0$. □

Corollary 4.8. *The chain decomposition of an element $a\in\mathbf{k}[Q]$ is unique, so the monomials $v_{G_1}^{\alpha_1}\cdots v_{G_n}^{\alpha_n}$ corresponding to all chains $G_1\supset\cdots\supset G_n$ and all exponents α_i form an additive basis of $\mathbf{k}[Q]$.*

The *f-vector* of Q is defined as $\boldsymbol{f}(Q)=(f_0,\dots,f_{n-1})$ where f_i is the number of faces of codimension $i+1$ (so that $f_0=m$ is the number of facets). The equivalent information is contained in the *h-vector* $\boldsymbol{h}(Q)=(h_0,\dots,h_n)$ determined from the equation

$$h_0t^n+\ldots+h_{n-1}t+h_n=(t-1)^n+f_0(t-1)^{n-1}+\ldots+f_{n-1}. \tag{4.9}$$

In particular, $h_0=1$ and $h_n=(-1)^n+(-1)^{n-1}f_0+\cdots+f_{n-1}$, which is equal to 1 when Q is face-acyclic.

Example 4.10. Let us consider the case $n=2$ described in Examples 2.4 and 3.2, so Q is a 2-ball with two 0-faces (say, p and q) and two 1-faces (say, G and H). Then $\boldsymbol{f}(Q)=(2,2)$, $\boldsymbol{h}(Q)=(1,0,1)$ and the face ring of Q is

$$\mathbf{k}[Q]=\mathbf{k}[v_G,v_H,v_p,v_q]/(v_Gv_H=v_p+v_q,\ v_pv_q=0),$$

where $\deg v_G=\deg v_H=2$, $\deg v_p=\deg v_q=4$.

The Poincaré series of the face ring looks exactly as in the classical case.

Theorem 4.11. *We have*

$$F\bigl(\mathbf{k}[Q];t\bigr)=\sum_{k=0}^{n}\frac{f_{k-1}t^{2k}}{(1-t^2)^k}=\frac{h_0+h_1t^2+\cdots+h_nt^{2n}}{(1-t^2)^n}.$$

Proof. By Corollary 4.8, in order to calculate the Poincaré series of $\mathbf{k}[Q]$ we need to calculate the number of monomials $v_{G_1}^{\alpha_1}\cdots v_{G_k}^{\alpha_k}$ with $\alpha_k \neq 0$ and $G_1 \supset \cdots \supset G_k$. First we note that the whole set $\mathcal{S}$ of such monomial splits into disjoint union of subsets consisting of monomials with the same last face in the chain. Let us restrict to one such subset, that is, fix a face G_k and consider only monomials whose last factor is $v_{G_k}^{\alpha_k}$ with $\alpha_k \neq 0$. Denote this subset by $\mathcal{S}_{G_k}$. Choose some vertex $p \subset G_k$ (it is unique if $k = n$ but otherwise we have a choice) and consider the restriction map s_p to the polynomial ring $\mathbf{k}[t_1, \dots, t_n]$. We may assume that $s_p(v_{G_k}) = t_1 \cdots t_k$, so that $s_p(\mathcal{S}_{G_k})$ actually lies in $\mathbf{k}[t_1, \dots, t_k]$. It is easy to see that $s_p(\mathcal{S}_{G_k})$ coincides with the set of monomials in $\mathbf{k}[t_1, \dots, t_k]$ divisible by $t_1 \cdots t_k$. Hence, the generating power series for monomials in $\mathcal{S}_{G_k}$ is $\frac{t^{2k}}{(1-t^2)^k}$. Since $\operatorname{codim} G_k = k$, the generating power series for the whole set $\mathcal{S}$ is exactly that given by the first identity in the theorem. The second identity is an obvious corollary of (4.9). □

4.2. **Simplicial posets.** The set of faces of a simplicial complex with empty set added is a poset (partially ordered set) with the empty set as the smallest element, and it is called the *face poset* of the simplicial complex. A poset $\mathcal{P}$ is called *simplicial* if it has a smallest element $\hat{0}$ and for each $x \in \mathcal{P}$ the lower segment $[\hat{0}, x]$ is a boolean algebra (the face poset of a simplex). The face poset of a simplicial complex is a simplicial poset, but there are simplicial posets that cannot be obtained in this way. Moreover, two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset is a more general notion than a simplicial complex and we will identify a simplicial complex with its face poset.

To each $x \in \overline{\mathcal{P}} := \mathcal{P} - \{\hat{0}\}$ we assign a geometrical simplex whose face poset is $[\hat{0}, x]$, and glue those geometrical simplices according to the order relation in $\mathcal{P}$. Then we get a cell complex such that the closure of each cell can be identified with a simplex preserving the face structure and all the attaching maps are inclusions. We call it a *simplicial cell complex* and denote its underlying space by $|\mathcal{P}|$. When $\mathcal{P}$ is (the face poset of) a simplicial complex K, then $|\mathcal{P}|$ agrees with the geometric realisation $|K|$ of K. The barycentric subdivision of a simplicial cell complex is obviously defined, and is again a simplicial cell complex.

Proposition 4.12. *The barycentric subdivision of a simplicial cell complex is a (geometric realisation of) simplicial complex.*

Proof. Indeed, we may identify the barycentric subdivision under question with the geometric realisation of the order complex $\Delta(\overline{\mathcal{P}})$ of the poset $\overline{\mathcal{P}}$. □

In the sequel we will not distinguish between simplicial posets and simplicial cell complexes, and call (the face poset of) the order complex $\Delta(\overline{P})$ the *barycentric subdivision* of $\mathcal{P}$. The set of faces of a nice manifold with corners Q forms a simplicial poset with respect to reversed inclusion (so Q itself is the smallest element). We call it the *face poset* of Q. It is a face poset of a simplicial complex if and only if all non-empty multiple intersections of facets of Q are connected.

Example 4.13. Take Q to be the orbit space S^{2n}/T of the torus manifold S^{2n} with the T-action in Example 2.4. There are n facets in Q and the intersection of any k facets is connected when $k \leqslant n-1$, but the intersection of n facets consists of two points. Therefore, the simplicial cell complex of the face poset of Q is obtained by gluing two $(n-1)$-simplices along their boundaries by the identity map. It is not a simplicial complex but (the geometric realisation of) it is homeomorphic to an $(n-1)$-sphere.

Let $\mathcal{P}$ be a simplicial poset. For each $x \in \overline{\mathcal{P}}$ set $\operatorname{rk} x = k$ if $[\hat{0}, x]$ is the face poset of a $(k-1)$-simplex. Introduce the polynomial ring $\mathbf{k}[v_x \colon x \in \overline{\mathcal{P}}]$ and make it graded by setting $\deg v_x = 2 \operatorname{rk} x$. We also write formally $v_{\hat{0}} = 1$. For every two elements $x, y \in \mathcal{P}$ denote by $x \vee y$ the set of their least common upper bounds, and by $x \wedge y$ the set of their greatest common lower bounds. Since $\mathcal{P}$ is simplicial, $x \wedge y$ consists of a single element provided that $x \vee y$ is non-empty. The following is an obvious dualisation of Definition 4.3.

Definition 4.14. The *face ring* of a simplicial poset $\mathcal{P}$ is the quotient

$$\mathbf{k}[\mathcal{P}] := \mathbf{k}[v_x \colon x \in \mathcal{P}]/\mathcal{I}_{\mathcal{P}},$$

where $\mathcal{I}_{\mathcal{P}}$ is the ideal generated by all

$$v_x v_y - v_{x \wedge y} \cdot \sum_{z \in x \vee y} v_z.$$

The notions of f-vector and h-vector can be defined for simplicial posets $\mathcal{P}$ ([15], [16]). In fact, when $\mathcal{P}$ is the face poset of a nice manifold with corners Q, we have $h_i(\mathcal{P}) = h_i(Q)$. There are also obvious analogues of Corollary 4.8 and Theorem 4.11.

The classical Stanley–Reisner face ring $\mathbf{k}[K]$ of a simplicial complex K is realized as the equivariant cohomology ring of a T-space by Davis and Januszkiewicz in [6], and it is not difficult to see that their argument works for a simplicial poset $\mathcal{P}$ as well. The order complex $\Delta(\overline{\mathcal{P}})$ is a simplicial complex. Let P be the cone of the geometric realisation $|\Delta(\overline{\mathcal{P}})|$ of $\Delta(\overline{\mathcal{P}})$. Since $|\Delta(\overline{\mathcal{P}})| = |\mathcal{P}|$, the "boundary" of P is $|\mathcal{P}|$. For each simplex $\sigma \in \Delta(\overline{\mathcal{P}})$, let $F_\sigma \subset P$ denote the geometric realisation of the poset $\{\tau \in \Delta(\overline{\mathcal{P}}) \colon \sigma \subset \tau\}$. If σ is a $(k-1)$-simplex, then we declare F_σ to be a *face of codimension* k. Thus, each facet (codimension-one face) can be identified with the star of some vertex in $\Delta(\overline{\mathcal{P}})$. Each codimension-$k$ face is a connected component of an intersection of k facets and acyclic since it is a cone. The space P with face decomposition was called in [6, p. 428] a simple polyhedral complex when $\mathcal{P}$ is a simplicial complex.

Suppose that the number of facets of P is m and that we have a map Λ in (3.4) satisfying the non-singular condition mentioned there. Then, the same construction as $M_Q(\Lambda)$ in (3.5) with Q replaced by P produces a T-space $M_P(\Lambda)$. This space may fail to be a manifold because P may not be a manifold, but the same argument as [6, Theorem 4.8] shows that $M_P(\Lambda)$ has the following nice property.

Proposition 4.15. *$H^*_T(M_P(\Lambda); \mathbf{k})$ is isomorphic to $\mathbf{k}[\mathcal{P}]$ as a ring.*

For a nice manifold with corners Q we take P to be the space associated with the face poset $\mathcal{P}$ of Q. Then there is a canonical equivariant map

$$\Phi\colon M_Q(\Lambda) \to M_P(\Lambda) \tag{4.16}$$

preserving the face structure. This is done inductively, starting from an identification of vertices and extending the map on each higher-dimensional face by a degree-one map. Every face of P is a cone, so there are no obstructions to such extensions. Since the map between orbit spaces preserves the face structure, it is covered by an equivariant map

$$M_Q(\Lambda) = T \times Q/\!\sim\, \to T \times P/\!\sim\, = M_P(\Lambda)$$

by the definition of identification spaces, see (3.5).

5. Axial functions and Thom classes

We investigate relationships between the equivariant cohomology of torus manifolds M and the face rings of their orbit spaces Q defined in the previous section. In fact, we show here that there is a natural ring homomorphism from $\mathbb{Z}[Q]$ to $H_T^*(M)$ modulo $H^*(BT)$-torsions and (in the next section) that it is an isomorphism when $H^{odd}(M) = 0$.

5.1. **Axial functions.** Like in the algebraic situation of the previous section, we have the restriction map to a sum of polynomial rings:

$$r = \bigoplus_{p \in M^T} r_p\colon H_T^*(M) \to H_T^*(M^T) = \bigoplus_{p \in M^T} H^*(BT). \tag{5.1}$$

The kernel of the map r is the $H^*(BT)$-torsion group, so it is injective when $H^{odd}(M) = 0$ by Lemma 2.1, but not surjective.

We identify M^T with the vertices of Q. The 1-skeleton of Q consisting of vertices (0-faces) and edges (1-faces) in Q forms an n-valent graph. Denote by $E(Q)$ the set of oriented edges in the 1-skeleton of Q. Let e be an element of $E(Q)$. The initial point and terminal point of e are denoted by $i(e)$ and $t(e)$ respectively. Then $M_e := \pi^{-1}(e)$ is a 2-sphere fixed by a codimension-one subtorus in T (here $\pi\colon M \to Q$ is the quotient map). It contains two T-fixed points $i(e)$ and $t(e)$. The 2-dimensional subspace $\mathcal{T}_{i(e)} M_e \subset \mathcal{T}_{i(e)} M$ is an irreducible component of the tangential T-representation $\mathcal{T}_{i(e)} M$. The same is true for the other T-fixed point $t(e)$ and the T-representations $\mathcal{T}_{i(e)} M$ and $\mathcal{T}_{t(e)} M$ are isomorphic. We want to view these representations as complex 1-dimensional representations and for that we need to assign orientations to them. We note that there is a unique characteristic submanifold, say M_i, such that it intersects M_e at $i(e)$ transversally. Since M and M_i are oriented by the definition of torus manifolds and they are even dimensional, a compatible orientation on the normal bundle of M_i, in particular, an orientation on $\mathcal{T}_{i(e)} M_e$, will be uniquely determined. The orientation on $\mathcal{T}_{i(e)} M_e$ determines a complex structure on it so that $\mathcal{T}_{i(e)} M_e$ can be viewed as a complex 1-dimensional T-representation. This defines an element of $\mathrm{Hom}(T, S^1) = H^2(BT)$, which we denote by $\alpha(e)$.

The normal bundle ν_i of M_i is an oriented T-vector bundle. Since the equivariant Euler class $e^T(\nu_i)$ lies in $H^2_T(M_i)$, its restriction to $p \in M_i^T$, denoted $e^T(\nu_i)|_p$, lies in $H^2_T(p) = H^2(BT)$. As is well known and easily checked, we have

$$e^T(\nu_i)|_p = \alpha(e), \tag{5.2}$$

where e is an oriented edge uniquely determined by these two conditions: $i(e) = p$ and $e \notin Q_i = \pi(M_i)$.

The function

$$\alpha\colon E(Q) \to H^2(BT).$$

(called an *axial function* in [10]) satisfies three properties:

1. $\alpha(-e) = \pm\alpha(e)$ for any $e \in E(Q)$, where $-e$ denotes e with opposite orientation.
2. At each vertex (or a T-fixed point) p, the set $\alpha_p := \{\alpha(e)\colon i(e) = p\}$ is a basis of $H^2(BT)$ over $\mathbb{Z}$.
3. For $e \in E(Q)$, we have $\alpha_{i(e)} \equiv \alpha_{t(e)} \mod \alpha(e)$.

The property 1 follows from the fact that $\mathcal{T}_{i(e)}M_e$ and $\mathcal{T}_{t(e)}M_e$ are isomorphic as real T-representations. In [10], the property $\alpha(-e) = -\alpha(e)$ is required in the definition of axial function but we allow $\alpha(-e) = \alpha(e)$. (Check that $\alpha(-e) = \alpha(e)$ for the standard T^2-action on S^4 in Example 2.4.) The property 2 follows from the fact that the T-representation $\mathcal{T}_{i(e)}M$ is faithful (and of complex dimension n). Property 3 follows from the fact that if T_e is the codimension one subtorus fixing M_e, then the T-representations $\mathcal{T}_{i(e)}M$ and $\mathcal{T}_{t(e)}M$ are isomorphic as T_e-representations because the points $i(e)$ and $t(e)$ are contained in M_e which is connected and fixed by T_e.

Lemma 5.3. *Let η be an element of $H^*_T(M)$. Then $r_{i(e)}(\eta) - r_{t(e)}(\eta)$ is divisible by $\alpha(e)$ for any $e \in E(Q)$.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc}
H^*_T(M) & \longrightarrow & H^*_T(i(e)) \oplus H^*_T(t(e)) = H^*(BT) \oplus H^*(BT) \\
\downarrow & & \downarrow \\
H^*_{T_e}(M_e) & \longrightarrow & H^*_{T_e}(i(e)) \oplus H^*_{T_e}(t(e)) = H^*(BT_e) \oplus H^*(BT_e)
\end{array}$$

where all the maps are restrictions. Since $H^*_{T_e}(M_e) = H^*(BT_e) \otimes H^*(M_e)$, the two components of the image of η in $H^*(BT_e) \oplus H^*(BT_e)$ by the above map coincide. Therefore it follows from the commutativity of the above diagram that the restrictions of $r_{i(e)}(\eta)$ and $r_{t(e)}(\eta)$ to $H^*(BT_e)$ coincide. Since the kernel of the restriction map $H^*(BT) \to H^*(BT_e)$ is the ideal generated by $\alpha(e)$, the lemma follows. □

5.2. **Thom classes.** The preimage $M_F := \pi^{-1}(F)$ of a codimension-k face $F \subset Q$ is a closed T-submanifold of M. It is a connected component of an intersection of k characteristic submanifolds. As remarked before, the prescribed orientations on M and characteristic submanifolds M_i determine compatible orientations on the normal bundles ν_i of M_i. These orientations determine an orientation on the normal bundle ν_F of M_F, hence on M_F because M is oriented. With this convention on orientations, we consider the Gysin homomorphism $H_T^0(M_F) \to H_T^{2k}(M)$ in the equivariant cohomology and denote the image of the identity element by τ_F. The element τ_F may be thought of as the Poincaré dual of M_F in the equivariant cohomology and called the *Thom class* of M_F. As is well known, the restriction image of $\tau_F \in H_T^{2k}(M)$ to $H_T^{2k}(M_F)$ agrees with the equivariant Euler class of ν_F and $r_p(\tau_F) = 0$ unless $p \in (M_F)^T$. It follows from (5.2) that

$$r_p(\tau_F) = \begin{cases} \prod\limits_{i(e)=p,\ e \not\subset F} \alpha(e), & \text{if } p \in (M_F)^T; \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

We set

$$\widehat{H}_T^*(M) := H_T^*(M)/H^*(BT)\text{-torsions.}$$

The restriction map r in (5.1) induces a monomorphism $\widehat{H}_T^*(M) \to H_T^*(M^T)$, which we also denote by r. The following lemma shows that the relations from Definition 4.3 hold in $\widehat{H}_T^*(M)$ with v_F replaced by τ_F.

Lemma 5.5. *For faces G and H of Q, the relations*

$$\tau_G \tau_H = \tau_{G\vee H} \cdot \sum_{E \in G\cap H} \tau_E,$$

hold in $\widehat{H}_T^*(M)$, where $\tau_\varnothing$ is understood to be 0.

Proof. Since the restriction map $r\colon \widehat{H}_T^*(M) \to H_T^*(M^T)$ is injective, it suffices to show that r_p maps both sides of the identity to the same element for every $p \in M^T$.

Let $p \in M^T$. For a face F, we set

$$N_p(F) := \{e \in E(Q)\colon i(e) = p,\ e \notin F\},$$

which may be thought of as the set of directions normal to F at p. Then the identity (5.4) can be written as

$$r_p(\tau_F) = \prod_{e \in N_p(F)} \alpha(e) \tag{5.6}$$

where the right hand side is understood to be zero if $N_p(F) = \varnothing$. If $p \notin G \cap H$, then $N_p(E) = \varnothing$ for any connected component E of $G\cap H$ and either $N_p(G) = \varnothing$ or $N_p(H) = \varnothing$. Therefore, both sides in the lemma map to zero by r_p. If $p \in G \cap H$, then

$$N_p(G) \cup N_p(H) = N_p(G \vee H) \cup N_p(E)$$

where E is the connected component of $G \cap H$ containing p, and $N_p(E') = \varnothing$ for any other connected component of $G \cap H$. This together with (5.6) shows that both sides in the lemma map to a same element by r_p. □

By virtue of the above lemma, the map $\mathbb{Z}[v_F \colon F \text{ a face}] \to H_T^*(M)$ sending v_F to τ_F induces a homomorphism

$$\varphi \colon \mathbb{Z}[Q] \to \widehat{H}_T^*(M). \tag{5.7}$$

Lemma 5.8. *The homomorphism φ is injective.*

Proof. We have $s = r \circ \varphi$, where s is the map from Lemma 4.6. Since s is injective, so is φ. □

6. Equivariant cohomology ring of torus manifolds with vanishing odd-degree cohomology

In this section we give a sufficient condition for the monomorphism φ in (5.7) to be an isomorphism (Theorem 6.5). In particular, it turns out that φ is an isomorphism when $H^{odd}(M) = 0$ (Corollary 6.6). Using this results, we give a description of the ring structure of $H^*(M)$ when $H^{odd}(M) = 0$ (Corollary 6.8).

6.1. Ring structure of equivariant cohomology.

The following theorem shows that the converse of Lemma 5.3 holds for torus manifolds with vanishing odd degree cohomology.

Theorem 6.1 ([8], see also Chapter 11 in [9])**.** *Suppose $H^{odd}(M) = 0$. For each $p \in M^T$, let η_p be an element of $H^*(BT)$. Then an element $(\eta_p) \in \bigoplus_{p \in M^T} H^*(BT)$ is in the image of the restriction map r in (5.1) if and only if $\eta_{i(e)} - \eta_{t(e)}$ is divisible by $\alpha(e)$ for any $e \in E(Q)$.*

Looking at the degree 0 cohomology, one sees from the theorem above that the 1-skeleton of Q is connected if $H^{odd}(M) = 0$. This holds for any face F of Q because $M_F = \pi^{-1}(F)$ is also a torus manifold with vanishing odd degree cohomology by Lemma 2.6. What we will use in the following argument is Lemma 5.3 and the connectedness of 1-skeletons of all faces of Q. (Actually, one can prove the connectedness of 1-skeletons without referring to the above theorem, see remark after Theorem 8.3.)

For a face F of Q, we denote by $I(F)$ the ideal in $H^*(BT)$ generated by all elements $\alpha(e)$ with $e \in F$.

Lemma 6.2. *Suppose that the 1-skeleton of a face F is connected and let η be an element of $H_T^*(M)$. Then, if $r_p(\eta) \notin I(F)$ for some vertex $p \in F$, then $r_q(\eta) \notin I(F)$ for any vertex $q \in F$.*

Proof. Suppose $r_q(\eta) \in I(F)$ for some vertex $q \in F$. Then $r_s(\eta) \in I(F)$ for any vertex $s \in F$ joined to q by an edge of F, say f, because $r_q(\eta) - r_s(\eta)$ is divisible by $\alpha(f)$ by Lemma 5.3. Since the 1-skeleton of F is connected, $\eta(q) \in I(F)$ for any vertex $q \in F$, which contradicts the assumption. □

Proposition 6.3. *Let M be a torus manifold with orbit space Q. If the 1-skeleton of every face of Q is connected, then $\widehat{H}_T^*(M)$ is generated by the elements τ_F as an $H^*(BT)$-module.*

Proof. Let $\eta \in H_T^{>0}(M)$ be a nonzero element. Set

$$Z(\eta) := \{p \in M^T \colon r_p(\eta) = 0\}.$$

Take $p \in M^T$ such that $p \notin Z(\eta)$. Then $r_p(\eta) \in H^*(BT)$ is non-zero and since the set $\{\alpha(e)\colon i(e) = p\}$ is a basis of $H^2(BT)$, $r_p(\eta)$ can be expressed as a polynomial in $\alpha(e)$'s with $i(e) = p$. Let

$$A \prod_{i(e)=p} \alpha(e)^{n_e} \tag{6.4}$$

be a monomial appearing in $r_p(\eta)$, where A is a non-zero integer and $n_e \geqslant 0$. Let F be the face spanned by the edges e with $n_e = 0$. Then $r_p(\eta) \notin I(F)$ since $r_p(\eta)$ contains the monomial (6.4). Hence, $r_q(\eta) \notin I(F)$, in particular $r_q(\eta) \neq 0$, for every vertex $q \in F$ by Lemma 6.2.

On the other hand, it follows from (5.4) that the monomial (6.4) can be written as $r_p(u_F \tau_F)$ with some $u_F \in H^*(BT)$. We consider an element $\eta' := \eta - u_F\tau_F \in H^*_T(M)$. Since $r_q(\tau_F) = 0$ for every vertex $q \notin F$, we have $r_q(\eta') = r_q(\eta)$ for such q. At the same time, $r_q(\eta) \neq 0$ for every vertex $q \in F$ (see above). It follows that $Z(\eta') \supset Z(\eta)$. However, the number of monomials in $r_p(\eta')$ is strictly smaller than that in $r_p(\eta)$. Therefore, subtracting a linear combination of τ_F's over $H^*(BT)$ from η we obtain an element λ such that $Z(\lambda)$ contains $Z(\eta)$ as a proper subset. Repeating this procedure, we end up at an element whose restriction to every vertex is zero. Since the restriction map $r\colon \widehat{H}^*_T(M) \to H^*_T(M^T)$ is injective, this finishes the proof. □

Theorem 6.5. *Let the situation be as in Proposition 6.3. Then the monomorphism* $\varphi\colon \mathbb{Z}[Q] \to \widehat{H}^*_T(M)$ *in (5.7) is an isomorphism.*

Proof. To prove that φ is surjective it suffices to show that $\widehat{H}^*_T(M)$ is generated by the elements τ_F as a ring. By Proposition 2.5, $\widehat{H}^2_T(M)$ is generated over $\mathbb{Z}$ by the elements τ_{Q_i} corresponding to the facets Q_i. (Note: the notation τ_i is used for τ_{Q_i} in Proposition 2.5.) In particular, any element in $H^2(BT) \subset \widehat{H}^*_T(M)$ can be written as a linear combination of τ_{Q_i}'s over $\mathbb{Z}$. Hence, any element in $H^*(BT)$ is a polynomial in τ_{Q_i}'s. The rest follows from Proposition 6.3. □

When $H^{odd}(M) = 0$, $H^*_T(M)$ is a free $H^*(BT)$-module by Lemma 2.1; so $\widehat{H}^*_T(M) = H^*_T(M)$.

Corollary 6.6. *For a torus manifold M with vanishing odd degree cohomology, the map* $\varphi\colon \mathbb{Z}[Q] \to H^*_T(M)$ *in (5.7) is an isomorphism.*

Proof. If $H^{odd}(M) = 0$, then the 1-skeleton of every face of Q is connected as remarked before. Therefore the corollary follows from Theorem 6.5. □

Remark. When $H^*(M)$ is generated in degree two, all non-empty multiple intersections of facets are connected by Lemma 2.8; so the face poset of Q is the face poset of the nerve of the covering of ∂Q, and since the nerve is a simplicial complex, $\mathbb{Z}[Q]$ reduces to the classical Stanley–Reisner face ring of the simplicial complex. Therefore, Corollary 6.6 is a generalisation of Proposition 3.4 in [13].

If $\mathcal{P}$ is the face poset of Q, then $\mathbb{Z}[\mathcal{P}] = \mathbb{Z}[Q]$ from the definition. The following is a characterisation of torus manifolds M with vanishing odd degree cohomology (or with cohomology generated in degree two) in terms of the face poset $\mathcal{P}$ associated with M.

Theorem 6.7. *Let M be a torus manifold with orbit space Q and let $\mathcal{P}$ be the face poset of Q. Then $H^{odd}(M)=0$ if and only if the following two conditions are satisfied:*

1. *$H^*_T(M)$ is isomorphic to $\mathbb{Z}[\mathcal{P}](=\mathbb{Z}[Q])$ as a ring, and*
2. *$\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay.*

Moreover, $H^*(M)$ is generated by its degree-two part if and only if $\mathcal{P}$ is (the face poset of) a simplicial complex in addition to the above two conditions.

Proof. If $H^{odd}(M)=0$, then $H^*_T(M)\cong\mathbb{Z}[Q]=\mathbb{Z}[\mathcal{P}]$ by Corollary 6.6, and $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay because $H^*_T(M)$ is a free $H^*(BT)$-module by Lemma 2.1. This proves the "only if" part in the former statement in the lemma.

We shall prove the "if" part. The composition of the homomorphisms

$$H^*(BT)\xrightarrow{\rho^*}H^*_T(M)\xrightarrow{r}\bigoplus_{p\in M^T}H^*(BT),$$

where $\rho\colon ET\times_T M\to BT$ is the projection, is the identity when restricted to each factor in the target, in other words, $r\circ\rho^*$ is a diagonal map. This implies that $\rho^*(t_1),\dots,\rho^*(t_n)$ is a linear system of parameters (an l.s.o.p.), see [2, Theorem 5.1.16]. By assumption $H^*_T(M)$ is isomorphic to $\mathbb{Z}[\mathcal{P}]$ and $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay, so every l.s.o.p. is a regular sequence (see [16, Theorem I.5.9]). It follows that $H^*_T(M)$ is a free $H^*(BT)$-module and hence $H^{odd}(M)=0$ by Lemma 2.1, proving the "if" part in the former statement in the theorem.

We shall prove the latter statement in the theorem. The "only if" part follows from Lemma 2.8 as remarked before. Suppose that $\mathcal{P}$ is the face poset of a simplicial complex. Then the face ring $\mathbb{Z}[\mathcal{P}]$ is generated by its degree-two part. Since $H^*_T(M)\cong\mathbb{Z}[\mathcal{P}]$ is a free $H^*(BT)$-module as proved above, $H^*(M)$ is a quotient ring of $H^*_T(M)$. It follows that $H^*(M)$ is also generated by its degree-two part. ∎

The following description of cohomology ring of a torus manifold with vanishing odd degree cohomology generalizes that of a complete non-singular toric variety, see [7, Proposition in p.106].

Corollary 6.8. *For a torus manifold M with vanishing odd degree cohomology,*

$$H^*(M)\cong\mathbb{Z}[v_F\colon F\ \text{a face of}\ Q]/I\qquad\text{as a ring,}$$

where I is the ideal generated by all

1. $v_G v_H - v_{G\vee H}\sum_{E\in G\cap H} v_E$;
2. $\sum_{i=1}^{m}\langle t,a_i\rangle v_{Q_i}$ *for* $t\in H^2(BT)$,

where Q_i are facets of Q and a_i are elements of $H_2(BT)$ in Proposition 2.5.

Proof. Since the Serre spectral sequence of the fibration $\rho\colon ET\times_T M\to BT$ collapses, the restriction map $H^*_T(M)\to H^*(M)$ is surjective and its kernel is the ideal generated by all $\rho(t)$ with $t \in H^{>0}(BT)$. However, since $H^*(BT)$ is a polynomial ring in degree two elements, it suffices to take degree two elements as t. Therefore, the corollary follows from Proposition 2.5 and Corollary 6.6. □

6.2. **Dehn-Sommerville equations.** Suppose that $H^{odd}(M) = 0$. Then, since $H_T^*(M) = H^*(BT) \otimes H^*(M)$ by Lemma 2.1 and $H^*(BT)$ is a polynomial ring in n variables of degree two, the Poincaré series of $H_T^*(M)$ is given by

$$F(H_T^*(M); t) = \frac{\sum_{i=0}^{n} \operatorname{rank}_{\mathbb{Z}} H^{2i}(M) t^{2i}}{(1-t^2)^n}.$$

On the other hand, the Poincaré series of the face ring $\mathbb{Z}[Q]$ is given in Theorem 4.11 and these two series must coincide by Corollary 6.6. It follows that

$$\operatorname{rank}_{\mathbb{Z}} H^{2i}(M) = h_i. \tag{6.9}$$

Since M is a manifold, the Poincaré duality implies that

$$h_i = h_{n-i}, \quad i = 0, \dots, n. \tag{6.10}$$

When every non-empty multiple intersection of facets in Q is connected, $\mathbb{Z}[Q]$ reduces to the classical Stanley–Reisner ring of the nerve of the covering of ∂Q and the equations (6.10) are the famous *Dehn–Sommerville equations* for the numbers of faces of the nerve simplicial complex.

7. Orbit spaces of torus manifolds with cohomology generated in degree two

From now on we discuss relations between the cohomology of a torus manifold M and the cohomology of its orbit space Q. The main result of this section is Theorem 7.3 which gives a cohomological characterisation of torus manifolds with orbit space a homology polytope. Using this result, we finally prove in the next section that Q is face-acyclic if $H^{odd}(M) = 0$. The following is a first step.

Lemma 7.1. *If $H^{odd}(M) = 0$, then $H^1(Q; \mathbf{k}) = 0$ for any coefficient $\mathbf{k}$, in particular, Q is orientable.*

Proof. We use the Leray spectral sequence (with $\mathbf{k}$ coefficient) of the projection map $ET \times_T M \to M/T = Q$ on the second factor. Its E_2 term is given by $E_2^{p,q} = H^p(X/T; \mathcal{H}^q)$ where $\mathcal{H}^q$ is a sheaf with stalk $H^q(BT_x; \mathbf{k})$ over a point $x \in X/T$, and the spectral sequence converges to $H_T^*(X; \mathbf{k})$. Since the T-action on M is locally standard by Theorem 3.3, the isotropy group T_x at $x \in M$ is a subtorus; so $H^{odd}(BT_x; \mathbf{k}) = 0$. Therefore we have $\mathcal{H}^{odd} = 0$, in particular, $\mathcal{H}^1 = 0$. Moreover, $\mathcal{H}^0 = \mathbf{k}$ (a constant sheaf). Therefore we have $E_2^{0,1} = 0$ and $E_2^{1,0} = H^1(X/T; \mathbf{k})$, whence $H^1(X/T; \mathbf{k}) \cong H_T^1(M; \mathbf{k})$. On the other hand, since $H^{odd}(M) = 0$ by assumption, $H_T^*(M)$ is a free $H^*(BT)$-module (isomorphic to $H^*(BT) \otimes H^*(M)$ by Lemma 2.1). Therefore, $H_T^{odd}(M; \mathbf{k}) = 0$ by the universal coefficient theorem. In particular, $H_T^1(M; \mathbf{k}) = 0$, thus proving the lemma. □

Lemma 7.2. *If either*

1. *Q is a homology polytope, or*

2. *$H^*(M)$ is generated by its degree-two part,*

then the face poset $\mathcal{P}$ of Q is (the face poset of) a simplicial Gorenstein complex, in particular, $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay and the geometric realisation $|\mathcal{P}|$ of $\mathcal{P}$ has the homology of an $(n-1)$-sphere.*

Proof. In both cases in the lemma, all non-empty multiple intersections of facets of Q are connected, so $\mathcal{P}$ agrees with the face poset of the nerve K of the covering of ∂Q where K is a simplicial complex. In the following we identify $\mathcal{P}$ with K.

First we prove that $\mathcal{P}$ is Gorenstein* in Case 1. According to Theorem II.5.1 of [16] it is enough to show that the link of a simplex σ of $\mathcal{P}$, denoted by $\operatorname{link}\sigma$, has the homology of a sphere of $\dim\operatorname{link}\sigma (= n-2-\dim\sigma)$. If $\sigma=\varnothing$ then $\operatorname{link}\sigma$ is $\mathcal{P}$ itself and its homology is isomorphic to the homology of the boundary ∂Q of Q because $\mathcal{P}$ is the nerve of Q, all faces of Q are acyclic and all non-empty multiple intersections of facets in Q are connected. If $\sigma\neq\varnothing$ then $\operatorname{link}\sigma$ agrees with the nerve of a face of Q. Since any face of Q is again a homology polytope, $\operatorname{link}\sigma$ has the homology of a sphere of $\dim\operatorname{link}\sigma$ by the same reasoning as the case $\sigma=\varnothing$.

Next we prove that $\mathcal{P}$ is Gorenstein* in Case 2. According to Theorem II.5.1 of [16] again, it is enough to show that (a) $\mathcal{P}$ is Cohen–Macaulay; (b) every $(n-2)$-dimensional simplex is contained in exactly two $(n-1)$-dimensional simplices; (c) $\chi(\mathcal{P})=\chi(S^{n-1})$. The condition (a) follows from Lemma 2.1 and Corollary 6.6. Let us prove (b). By definition, every k-dimensional simplex of $\mathcal{P}$ corresponds to a set of $k+1$ characteristic submanifolds having non-empty intersection. By Lemma 2.8, the intersection of any n characteristic submanifolds is either empty or consists of a single T-fixed point. This means that the $(n-1)$-simplices of $\mathcal{P}$ are in one-to-one correspondence with the T-fixed points of M. Now, each $(n-2)$-simplex of $\mathcal{P}$ corresponds to a non-empty intersection of $n-1$ characteristic submanifolds of M. The latter is connected again by Lemma 2.8 and has a non-trivial T-action, so it is a 2-sphere. Every 2-sphere contains exactly two T-fixed points, and this implies (b). Finally, (c) is just the Dehn–Sommerville equation $h_0=h_n$, see the sentence following (4.9). ◻

Theorem 7.3. *The cohomology of a torus manifold M is generated by its degree-two part if and only if M is locally standard and the orbit space Q is a homology polytope.*

Proof. Let $\mathcal{P}$ be the face poset of Q and P be the space with face structure associated with $\mathcal{P}$, see Subsection 4.2. As a space, P is the cone of $|\mathcal{P}|$.

We first prove the "if" part. Suppose Q is a homology polytope. Since $H^2(Q)=0$, M is equivariantly homeomorphic to $M_Q(\Lambda)$ by Lemma 3.6; so we may regard the map Φ in (4.16) as a map from M to M_P. Let $M_{P,i}$ be characteristic subcomplexes of $M_P=M_P(\Lambda)$ defined similarly to characteristic submanifolds M_i of M. Since the T-actions on $M_P\setminus\cup_i M_{P,i}$ and $M\setminus\cup_i M_i$ are free, we have

$$H_T^*(M_P,\cup_i M_{P,i})\cong H^*(P,|\mathcal{P}|),\quad H_T^*(M,\cup_i M_i)\cong H^*(Q,\partial Q).$$

Therefore, the map Φ induces a map between exact sequences

$$
\begin{array}{ccccccc}
\longrightarrow & H^*(P,|\mathcal{P}|) & \longrightarrow & H_T^*(M_P) & \longrightarrow & H_T^*(\cup_i M_{P,i}) & \longrightarrow \\
 & \big\downarrow & & \big\downarrow \Phi^* & & \big\downarrow & \\
\longrightarrow & H^*(Q,\partial Q) & \longrightarrow & H_T^*(M) & \longrightarrow & H_T^*(\cup_i M_i) & \longrightarrow
\end{array}
\tag{7.4}
$$

Each M_i itself is a torus manifold over a homology polytope Q_i. Using induction and a Mayer–Vietoris argument, we may assume that the map $H_T^*(\cup_i M_{P,i}) \to H_T^*(\cup_i M_i)$ above is an isomorphism. By Lemma 7.2 $|\mathcal{P}|$ has the homology of an $(n-1)$-sphere and since P is the cone of $|P|$, we have $H^*(P,|\mathcal{P}|) \cong H^*(D^n, S^{n-1})$. We also have $H^*(Q,\partial Q) \cong H^*(D^n, S^{n-1})$ because Q is a homology polytope. Through these isomorphisms, we see from the construction of the map Φ that the induced map $H^*(P,|\mathcal{P}|) \to H^*(Q,\partial Q)$ is the identity map on $H^*(D^n, S^{n-1})$. Therefore, the 5-lemma applied to (7.4) shows that $\Phi^*\colon H_T^*(M_P) \to H_T^*(M)$ is an isomorphism; whence $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}]$ by Proposition 4.15. We also know that $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay by Lemma 7.2. Therefore, the two conditions in Theorem 6.7 are satisfied. Moreover, since Q is a homology polytope, $\mathcal{P}$ agrees with the face poset of the nerve simplicial complex of the covering of ∂Q. It follows that $H^*(M)$ is generated by its degree-two part by Theorem 6.7, which finishes the proof of the "if" part.

Now we prove the "only if" part. Suppose that $H^*(M)$ is generated by the degree-two elements. Then M is locally standard by Theorem 3.3. Since all non-empty multiple intersections of characteristic submanifolds are connected and their cohomology rings are generated in degree two by Lemma 2.8, we may assume by induction that all the proper faces of Q are homology polytopes, in particular they are acyclic, whence $H^*(\partial Q) \cong H^*(|\mathcal{P}|)$. This together with Lemma 7.2 shows that

$$H^*(\partial Q) \cong H^*(S^{n-1}). \tag{7.5}$$

Claim. $H^2(Q) = 0$.

The claim is trivial for $n = 1$. If $n = 2$ then Q is a surface with boundary, hence, $H^2(Q) = 0$ in this case too. Now assume $n \geqslant 3$. Let us consider the exact equivariant cohomology sequence of pair $(M, \cup_i M_i)$, see (7.4). All the maps in the exact sequence are $H^*(BT)$-module maps. By Lemma 2.1, $H_T^*(M)$ is a free $H^*(BT)$-module. On the other hand, $H^*(Q,\partial Q)$ is finitely generated over $\mathbb{Z}$, so it is a torsion $H^*(BT)$-module. It follows that the whole sequence splits in short exact sequences:

$$0 \to H_T^k(M) \to H_T^k(\cup_i M_i) \to H^{k+1}(Q,\partial Q) \to 0 \tag{7.6}$$

Taking $k = 1$ above, we get

$$H_T^1(\cup_i M_i) \cong H^2(Q,\partial Q).$$

Here the same argument as in Lemma 7.1 shows that the former is isomorphic to $H^1((\cup_i M_i)/T) = H^1(\partial Q)$, and the above isomorphism implies (through the projection $(ET \times M)/T \to M/T = Q$) that the coboundary map $H^1(\partial Q) \to$

$H^2(Q,\partial Q)$ in the exact sequence of the pair $(Q,\partial Q)$ is an isomorphism. Therefore, we get the following exact sequence fragment:

$$0\to H^2(Q)\to H^2(\partial Q)\to H^3(Q,\partial Q).$$

Since $H^2(\partial Q)\cong H^2(S^{n-1})$ by (7.5), we have $H^2(Q)=0$ if $n\geqslant 4$. When $n=3$, the coboundary map above is an isomorphism because Q is orientable by Lemma 7.1, whence $H^2(Q)=0$ again. This completes the proof of the claim.

Since $H^2(Q)=0$, we have a map $\Phi\colon M\to M_P(\Lambda)$ similarly to the "if" part proof above. Let us consider the diagram (7.4) with $\mathbf{k}$ coefficient where $\mathbf{k}=\mathbb{Q}$ or $\mathbb{Z}/p$ with prime p. Using induction and a Mayer–Vietoris argument, we deduce that $H^*_T(\cup_i M_{P,i};\mathbf{k})\to H^*_T(\cup_i M_i;\mathbf{k})$ is an isomorphism. We know that $H^*(P,|\mathcal{P}|;\mathbf{k})\cong H^*(D^n,S^{n-1};\mathbf{k})$ by Lemma 7.2, and it follows from the construction of Φ that the homomorphism

$$H^*(D^n,S^{n-1};\mathbf{k})\cong H^*(P,|\mathcal{P}|;\mathbf{k})\to H^*(Q,\partial Q;\mathbf{k}) \tag{7.7}$$

induced from Φ is an isomorphism on degree n, thus injective on all degrees. Therefore (an extended version of) the 5-lemma (see [14, p.185]) applied to (7.4) with $\mathbf{k}$ coefficient shows that $\Phi^*\colon H^*_T(M_P;\mathbf{k})\to H^*_T(M;\mathbf{k})$ is injective. Here, $H^*_T(M)\cong\mathbb{Z}[Q]\cong H^*_T(M_P)$ by Corollary 6.6 (or Proposition 3.4 in [13]) and Proposition 4.15 (or Theorem 4.8 of [6]), so $H^*_T(M_P;\mathbf{k})$ and $H^*_T(M;\mathbf{k})$ have the same dimension over $\mathbf{k}$ on each degree. Therefore, the monomorphism $\Phi^*\colon H^*_T(M_P;\mathbf{k})\to H^*_T(M;\mathbf{k})$ is actually an isomorphism. Again, the 5-lemma applied to (7.4) with $\mathbf{k}$ coefficient implies that the map (7.7) is an isomorphism, so $H^*(Q,\partial Q;\mathbf{k})\cong H^*(D^n,S^{n-1};\mathbf{k})$ for any $\mathbf{k}$ and hence $H^*(Q,\partial Q)\cong H^*(D^n,S^{n-1})$. This together with (7.5) (or the Poincaré-Lefschetz duality) gives the acyclicity of Q, thus finishing the proof of the theorem. ◻

The following characterises simplicial complexes associated with torus manifolds with cohomology generated in degree two.

Theorem 7.8. *A simplicial complex $\mathcal{P}$ is associated with a torus manifold M whose cohomology is generated by its degree-two part if and only if $\mathcal{P}$ is Gorenstein* and $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p.*

Proof. If $H^*(M)$ is generated by its degree-two part, then $\mathcal{P}$ is Gorenstein*, in particular $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay Lemma 7.2. Moreover $H^*_T(M)\cong\mathbb{Z}[\mathcal{P}]$ by Corollary 6.6 (or Proposition 3.4 in [13]) and since $H^*_T(M)\cong H^*(BT)\otimes H^*(M)$ as an $H^*(BT)$-module by Lemma 2.1, $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p.

Now we prove the "if" part. According to Theorem 12.2 of [5], there exists a homology polytope Q whose nerve is $\mathcal{P}$. As usual, let $\mathbb{Z}[\mathcal{P}]$ be the quotient of the polynomial ring $\mathbb{Z}[v_1,\dots,v_m]$ by the Stanley-Reisner ideal $\mathcal{I}_{\mathcal{P}}$. Since $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p., it is a free module over a polynomial ring $\mathbb{Z}[t_1,\dots,t_n]$ in n variables. Since there is no degree two element in the ideal $\mathcal{I}_{\mathcal{P}}$, a linear combination t of $t_1,\dots,t_n$ over $\mathbb{Z}$, which is an element of $\mathbb{Z}[\mathcal{P}]$, can be uniquely expressed as

$$t=\sum_{i=1}^m a_i(t)v_i$$

with integers $a_i(t)$ depending on t. Clearly, $a_i(t)$ is linear on t, so a_i can be viewed as an element of the dual space of the span of $t_1, \dots, t_n$ over $\mathbb{Z}$ (cf. Proposition 2.5). The dual space is isomorphic to $\mathbb{Z}^n \cong H_2(BT)$ and we define Λ to be a map sending i to a_i. Then $M := M_Q(\Lambda)$ is a torus manifold with cohomology generated by its degree-two elements by Theorem 7.3, thus the required torus manifold. □

8. Orbit spaces of torus manifolds with vanishing odd degree cohomology

Let F be a face of Q. The facial submanifold $M_F = \pi^{-1}(F)$ is a connected component of an intersection of finitely many characteristic submanifolds. The Whitney sum of normal bundles to these characteristic submanifolds restricted to M_F gives the normal bundle ν_F of M_F. As remarked before, the prescribed orientations on M and its characteristic submanifolds determine a T-invariant complex structure on the normal bundles of characteristic submanifolds. These determine a T-invariant complex structure on ν_F so that the complex projective bundle $P(\nu_F)$ of ν_F can be considered. Replacing M_F in M by $P(\nu_F)$, one obtains a new torus manifold $\widetilde{M}$. The passage from M to $\widetilde{M}$ is called the *blowing-up* of M at M_F. (Remark: The normal bundle ν_F admits many invariant complex structures and the following argument works once we choose one.) The orbit space $\widetilde{Q}$ of $\widetilde{M}$ is then the result of "cutting" Q along the face F, and the simplicial cell complex dual to $\widetilde{Q}$ is obtained from that dual to Q by applying a stellar subdivision of the face dual to F.

Lemma 8.1. *The orbit space $\widetilde{Q}$ is face-acyclic if and only if so is Q.*

Proof. By cutting Q along the face F one produces a new facet $\widetilde{F} \subset \widetilde{Q}$ that contains all other new proper faces of $\widetilde{Q}$. The projection map $\widetilde{Q} \to Q$ collapses $\widetilde{F}$ back to F. The face F is a deformation retract of $\widetilde{F}$. Hence, F is acyclic if and only if $\widetilde{F}$ is acyclic. Same is true for any other new face of $\widetilde{Q}$. It is also clear from the construction that Q is a deformation retract of $\widetilde{Q}$. Therefore, $\widetilde{Q}$ is acyclic if and only if so is Q. □

Lemma 8.2. $H^{odd}(\widetilde{M}) = 0$ *if* $H^{odd}(M) = 0$.

Proof. The facial submanifold $M_F \subset M$ is blown up to a codimension-two facial submanifold $\widetilde{M}_{\widetilde{F}} \subset \widetilde{M}$, in fact, $\widetilde{M}_{\widetilde{F}} = P(\nu_F)$. Since $\widetilde{M}_{\widetilde{F}}$ is the total space of a bundle with base M_F and fibre a complex projective space, its cohomology is a free $H^*(M_F)$-module on even-dimensional generators by Dold's theorem (see, e.g., [17, Ch. V]). If $H^{odd}(M) = 0$, then $H^{odd}(M_F) = 0$ by Lemma 2.6 and hence $H^{odd}(\widetilde{M}_{\widetilde{F}}) = 0$. Let $\widetilde{M} \to M$ be the collapse map and consider the diagram

$$\begin{array}{ccccccc}
H^{k-1}(M_F) & \longrightarrow & H^k(M, M_F) & \longrightarrow & H^k(M) & \longrightarrow & H^k(M_F) \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\
H^{k-1}(\widetilde{M}_{\widetilde{F}}) & \longrightarrow & H^k(\widetilde{M}, \widetilde{M}_{\widetilde{F}}) & \longrightarrow & H^k(\widetilde{M}) & \longrightarrow & H^k(\widetilde{M}_{\widetilde{F}})
\end{array}$$

where the second vertical arrow is an isomorphism by excision. Assume that k is odd. If $H^{odd}(M)=0$ then $H^{k-1}(M_F)\to H^k(M,M_F)$ is onto. Therefore, it follows from the above commutative diagram that $H^{k-1}(\widetilde{M}_{\widetilde{F}})\to H^k(\widetilde{M},\widetilde{M}_{\widetilde{F}})$ is also onto. Since $H^k(\widetilde{M}_{\widetilde{F}})=0$, this implies $H^k(\widetilde{M})=0$. □

The following main result of this section is an analogue of Theorem 7.3.

Theorem 8.3. *The odd-degree cohomology of M vanishes if and only if M is locally standard and the orbit space Q is face-acyclic.*

Proof. The idea is to reduce to Theorem 7.3 by blowing up sufficiently many facial submanifolds $M_F=\pi^{-1}(F)$. Since the barycentric subdivision is a sequence of stellar subdivisions, by applying sufficiently many blow-ups one gets a torus manifold $\widehat{M}$ with orbit space $\widehat{Q}$ such that the face poset of $\widehat{Q}$ is the barycentric subdivision of the face poset of Q. The collapse map $\widehat{M}\to M$ is decomposed into a sequence of collapse maps for single blow-ups:

$$M=M_0 \longleftarrow M_1 \longleftarrow \dots \longleftarrow M_k=\widehat{M}. \tag{8.4}$$

We first prove the "only if" part. Assume that $H^{odd}(M)=0$. Then M is locally standard by Theorem 3.3. By applying Lemma 8.2 several times we get $H^{odd}(\widehat{M})=0$. By construction, all the intersections of faces of $\widehat{Q}$ are connected, so $H^*(\widehat{M})$ is generated by its degree-two part by Theorem 6.7 and $\widehat{Q}$ is a homology polytope by Theorem 7.3, in particular, face-acyclic. Finally, by applying Lemma 8.1 inductively we conclude that Q is also face-acyclic.

The scheme of the proof of the "if" part is same as that of Theorem 7.3. But there are two things to be checked. Those are

1. $|\mathcal{P}|$ has the homology of an $(n-1)$-sphere,
2. $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay.

Let $\widehat{\mathcal{P}}$ be the face poset of $\widehat{Q}$. Since Q is face-acyclic, $\widehat{Q}$ is a homology polytope. Therefore, $|\widehat{\mathcal{P}}|$ has the homology of an $(n-1)$-sphere by Lemma 7.2. However, $|\widehat{\mathcal{P}}|=|\mathcal{P}|$, so the first statement above follows. Since $\widehat{Q}$ is a homology polytope, $\mathbb{Z}[\widehat{\mathcal{P}}]$ is Cohen-Macaulay by Lemma 7.2. This implies that $\mathbb{Z}[\mathcal{P}]$ itself is Cohen-Macaulay by Corollary 3.7 of [15], proving the second statement above. □

Remark. As one can easily observe, the argument in the "only if" part of the above theorem is independent of Theorem 6.1 and Corollary 6.6. Now, given that Q is face-acyclic, one readily deduces that the 1-skeleton of Q is connected. Indeed, otherwise the smallest face containing vertices from two different connected components of the 1-skeleton would be a manifold with at least two boundary components and thereby non-acyclic. Thus, our reference to Theorem 6.1 above was actually irrelevant, although it makes the arguments more straightforward.

9. Betti numbers of torus manifolds with vanishing odd degree cohomology

The barycentric subdivision $\widehat{\mathcal{P}}$ of a simplicial poset $\mathcal{P}$ is (the face poset of) a simplicial complex and $\mathcal{P}$ is said to be Gorenstein* if $\widehat{\mathcal{P}}$ is Gorenstein* ([15],

[16]). If $\mathcal{P}$ is the simplicial poset associated with a torus manifold M with vanishing odd degree cohomology, then the torus manifold $\widehat{M}$ corresponding to $\widehat{\mathcal{P}}$ has cohomology generated by its degree-two part as remarked in the proof of Theorem 8.3. Hence, $\widehat{\mathcal{P}}$ is Gorenstein* by Lemma 7.2 and $\mathcal{P}$ is Gorenstein* by definition. In [15] Stanley proved that any vector satisfying the conditions in Theorem 9.2 below is an h-vector of a Gorenstein* simplicial poset. He also conjectured that those conditions are necessary. In this section we prove his conjecture for Gorenstein* simplicial posets $\mathcal{P}$ associated with torus manifolds M with vanishing odd degree cohomology, and characterize h-vectors of those Gorenstein* simplicial posets. Since

$$h_i(\mathcal{P}) = \operatorname{rank}_{\mathbb{Z}} H^{2i}(M), \tag{9.1}$$

by (6.9), our problem is equivalent to characterizing Betti numbers of torus manifolds with vanishing odd degree cohomology. We note that

$$h_i(\mathcal{P}) \geqslant 0,\ h_i(\mathcal{P}) = h_{n-i}(\mathcal{P}) \text{ for any } i, \text{ and } h_0(\mathcal{P}) = 1.$$

Theorem 9.2. *Let $(h_0, h_1, \dots, h_n)$ be a vector of non-negative integers with $h_i = h_{n-i}$ for any i and $h_0 = 1$. Any of the following (mutually exclusive) conditions are necessary and sufficient for the existence of a Gorenstein* simplicial poset $\mathcal{P}$ of rank n such that $\mathcal{P}$ is associated with a torus manifold of dimension $2n$ with vanishing odd degree cohomology and $h_i(\mathcal{P}) = h_i$ for any i:*

1. *n is odd,*
2. *n is even and $h_{n/2}$ is even,*
3. *n is even, $h_{n/2}$ is odd, and $h_i > 0$ for any i.*

Proof. For a torus manifold M, we set $h_i(M) = \operatorname{rank}_{\mathbb{Z}} H^{2i}(M)$. Thanks to (9.1), we may use $h_i(M)$ instead of $h_i(\mathcal{P})$ to prove the theorem.

We shall prove the sufficiency first. Examples 2.3 and 2.4 produce torus manifolds $\mathbb{C}P^n$ and $S^{2n-2k} \times S^{2k}$ for $0 \leqslant k \leqslant n$. In both cases the odd-degree cohomology is zero. If M_1 and M_2 are torus manifolds (of same dimension) with vanishing odd degree cohomology, then their equivariant connected sum $M_1 \# M_2$ at fixed points (having isomorphic tangential representations) produces a torus manifold with vanishing odd degree cohomology and

$$h_i(M_1 \# M_2) = h_i(M_1) + h_i(M_2) \qquad \text{for } 1 \leqslant i \leqslant n-1.$$

Therefore, taking equivariant connected sum of $\mathbb{C}P^n$, S^{2n} and $S^{2n-2k} \times S^{2k}$ for $1 \leqslant k \leqslant n-1$ one easily gets any vector satisfying the conditions in the theorem.

Now we prove the necessity. Let M be a torus manifold of dimension $2n$. It suffices to prove that $h_{n/2}(M)$ is even if n is even and $h_i(M) = 0$ for some $i > 0$.

Let G be the 2-torus subgroup of T of rank n. Then the equivariant total Stiefel–Whitney class of M with the restricted G-action is defined to be the ordinary total Stiefel–Whitney class of the vector bundle $EG \times_G \mathcal{T}M \to EG \times_G M$, and is denoted by $w^G(M)$. By definition, $w^G(M)$ lies in $H^*_G(M; \mathbb{Z}/2)$. We denote by τ_i the image of the identity by the equivariant Gysin map

$H^0_T(M_i;\mathbb{Z}/2) \to H^2_T(M;\mathbb{Z}/2)$, where M_i $(i=1,\dots,m)$ are characteristic submanifolds of M as before.

Claim. $w^G(M) = \prod_{i=1}^m (1+\tau_i)$.

The proof of the claim is essentially same as that of Theorem 3.1 in [13], where the same formula was proved for total equivariant Chern class, but for the reader's convenience we shall give a proof. Since $H^{odd}(M;\mathbb{Z}/2)=0$ and $M^G = M^T$ is isolated, we have

$$\dim H^*(M;\mathbb{Z}/2) = \chi(M) = \chi(M^T) = \chi(M^G) = \dim H^*(M^G;\mathbb{Z}/2).$$

Therefore, $H^*_G(M;\mathbb{Z}/2)$ is a free $H^*(BG;\mathbb{Z}/2)$-module (see [1, Theorem 1.6 in p.374]). It follows from the localization theorem that the restriction map

$$H^*_G(M;\mathbb{Z}/2) \to H^*_G(M^G;\mathbb{Z}/2) \tag{9.3}$$

is injective. For $p \in M^G = M^T$, set $I(p) := \{i\colon p \in M_i\}$. The cardinality of $I(p)$ is n and the tangential G-representation $\mathcal{T}_pM$ decomposes as

$$\mathcal{T}_pM = \bigoplus_{i\in I(p)} \nu_i|_p$$

where ν_i is the normal bundle of M_i to M and $\nu_i|_p$ is its restriction to p. It follows that

$$w^G(M)|_p = \prod_{i\in I(p)} w^G(\nu_i|_p). \tag{9.4}$$

Here, since ν_i is orientable and of real dimension two, $w_1^G(\nu_i)=0$ and $w_2^G(\nu_i)$ is the mod 2 reduction of the equivariant Euler class of ν_i. Therefore, we have $w_2^G(\nu_i|_p) = \tau_i|_p$ for $i \in I(p)$. Moreover, $\tau_i|_p = 0$ for $i \notin I(p)$ by a property of equivariant Gysin homomorphism. Thus the identity (9.4) turns into

$$w^G(M)|_p = \prod_{i\in I(p)} (1+\tau_i)|_p = \prod_{i=1}^m (1+\tau_i)|_p.$$

This together with the injectivity of the restriction map in (9.3) proves the claim.

Through the map $H^*_G(M;\mathbb{Z}/2) \to H^*(M;\mathbb{Z}/2)$ obtained by forgetting the G-action, the equivariant Stiefel–Whitney class $w^G(M)$ reduces to the (ordinary) Stiefel–Whitney class $w(M)$ of M. Since τ_i is of degree two, the claim above shows that $w_{2n}(M)$ is a polynomial in degree two elements. Assume $h_i(M)=0$ for some $i>0$. Then $w_{2n}(M)=0$. The mod 2 reduction of the Euler characteristic $\chi(M)$ of M agrees with $w_{2n}(M)$ evaluated on the mod 2 fundamental class of M, so the vanishing of $w_{2n}(M)$ means that $\chi(M)$ is even. Here $\chi(M) = \sum_{i=0}^n h_i(M)$ and $h_i(M) = h_{n-i}(M)$ by Poincaré duality, thus $h_{n/2}(M)$ must be even for even n. □

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