Direct images of $\mathcal{D}$-modules in prime characteristic *

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Last year two remarkable results appeared concerning the $\mathcal{D}$-modules on the flag variety over an algebraically closed field $\Bbbk$ of chracteristic $p > 0$. One was due to Kashiwara M. and N. Lauritzen [KLa02] showing the failure of $\mathcal{D}$-affinity of the flag variety in SL_5, and the other by R. Bezrukavnikov, I. Mirkovic and D. Rumynin [BMR]; they establish instead a derived equivalence between the category of finite generated modules over the universal enveloping algebra of the Lie algebra of the relevant simple algebraic group G having the trivial Harish-Chandra character and the category of coherent modules over the sheaf of rings of crystalline differential operators on the flag variety, and succeds in computing the number of irreducibles for the Lie algebra with a fixed Frobenius central character. On any smooth $\Bbbk$-scheme X their crystalline differential operators are just the 0-th term of Berthelot's rings $\mathcal{D}_X^{(m)}$, $m \in \mathbb{N}$, of arithmetic differential operators [B96]. Those $\mathcal{D}_X^{(m)}$'s form a direct system whose direct limit is the usual sheaf $\mathcal{D}iff_X$ of differential operators. The images $\bar{\mathcal{D}}_X^{(m)}$ of $\mathcal{D}_X^{(m)}$ in $\mathcal{D}iff_X$ form the p-filtration of $\mathcal{D}iff_X$ studied by B. Haarstert [H88].

In this note we will clarify a relashionship of $\mathcal{D}_X^{(m)}$ and $\bar{\mathcal{D}}_X^{(m)}$ with respect to direct image functors, and construct on the flag variety a $\bar{\mathcal{D}}^{(m)}$-module, whose global sections constitute a standard module for the $(m+1)$-st Frobenius kernel of G. That $\bar{\mathcal{D}}^{(m)}$-module is supported by a point, and is a unique irreducible $\bar{\mathcal{D}}^{(m)}$-module having the same support.

An advantage of $\mathcal{D}^{(m)}$ over $\bar{\mathcal{D}}^{(m)}$ is that $\mathcal{D}^{(m)}$ is defined over the ring of p-adic integers $\mathbb{Z}_p$. Thus a theory of $\mathcal{D}^{(m)}$-modules over $\mathbb{Z}_p$ on the flag variety invites our exploration.

If X is a scheme, by $\mathbf{Mod}_X$ (resp. $\mathcal{M}od_X$, $\otimes_X$) we will mean $\mathbf{Mod}_{\mathcal{O}_X}$ (resp. $\mathcal{M}od_{\mathcal{O}_X}$, $\otimes_{\mathcal{O}_X}$).

1° Crystalline differential operators

(1.1) Let G be a simply connected simple algebraic group over an algebraically closed field $\Bbbk$, $\Bbbk[G]$ the Hopf algebra defining G, $\varepsilon_G : \Bbbk[G] \to \Bbbk$ the counit of $\Bbbk[G]$, $\mathfrak{m}_G = \ker(\varepsilon_G)$, and $\mathrm{Dist}(G) = \{\mu \in \Bbbk[G]^* \mid \mu(\mathfrak{m}_G^{n+1}) = 0\ \exists n \in \mathbb{N}\}$ the algebra of distributions on G. Denote the Lie algebra $(\mathfrak{m}_G/\mathfrak{m}_G^2)^* \subseteq \mathrm{Dist}(G)$ of G by $\mathfrak{g}$ and by $\mathbf{U}$ its universal enveloping algebra.

If $\mathbf{U}_{\mathbb{Z}}$ is Kostant's $\mathbb{Z}$-form of the universal enveloping algebra over $\mathbb{C}$ of the simple

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$\mathbb{C}$-Lie algebra of the same type as $\mathfrak{g}$, there is an isomorphism of $\Bbbk$-algebras

$$\mathrm{Dist}(G) \simeq \mathbf{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk.$$

A finite dimensional G-module is naturally a $\mathrm{Dist}(G)$-module, and vice versa.

Let B be a Borel subgroup of G, $\mathcal{B} = G/B$ the flag variety of G, and $\mathcal{D}iff = \mathcal{D}iff_{\mathcal{B}/\Bbbk}$ the sheaf of $\Bbbk$-algebras of differential operators on $\mathcal{B}$ as defined in [EGAIV]. In positive characteristic the Beilinson-Bernstein localization theorem [BB81] fails:

Theorem: *Assume* $\mathrm{ch}\,\Bbbk > 0$.

(i) **Smith [Sm86]**: *The $\Bbbk$-algebra homomorphism*

$$\mathrm{Dist}(G) \to \Gamma(\mathcal{B}, \mathcal{D}iff)$$

induced by the G-equivariant structure on $\mathcal{O}_{\mathcal{B}}$ is not surjective in SL_2.

(ii) **Kashiwara-Lauritzen [KLa02]:** *In SL_5 there is a quasi-coherent $\mathcal{D}iff$-module $\mathcal{M}$ of finite type such that*

$$\mathrm{H}^1(\mathcal{B}, \mathcal{M}) \neq 0.$$

Throughout the rest of the manuscript we assume unless otherwise specified that $\Bbbk$ has positive characteristic p.

(1.2) Instead of $\mathrm{Dist}(G)$ and $\mathcal{D}iff$, Bezrukavnikov, Mirkovic and Rumynin [BMR] consider the universal enveloping algebra $\mathbf{U}$ and the sheaf $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$ of $\Bbbk$-algebras of crystalline differential operators on $\mathcal{B}$ introduced by [BB93]:

$$\mathcal{D} = \mathrm{T}_{\Bbbk}(\mathcal{D}iff^1)/ \\ (\lambda - \lambda 1_{\mathcal{O}_{\mathcal{B}}}, a \otimes \delta - a\delta, \delta \otimes \delta' - \delta' \otimes \delta - [\delta, \delta'] \mid \lambda \in \Bbbk, a \in \mathcal{O}_{\mathcal{B}}; \delta, \delta' \in \mathcal{D}iff^1),$$

where $\mathcal{D}iff^1$ is the sheaf of differential operators of order ≤ 1 in $\mathcal{D}iff$ and $\mathrm{T}_{\Bbbk}(\mathcal{D}iff^1)$ is the tensor algebra over $\Bbbk$ of $\mathcal{D}iff^1$. In charactristic 0 one has $\mathcal{D} \simeq \mathcal{D}iff$.

To describe the work [BMR], assume for simplicity in the rest of §1 that $p > 2(h-1)$, h the Coxeter number of G. Let T be a maximal torus of B and $\Lambda = \mathbf{GrpSch}(T, GL_1)$. We will write the group operation on Λ additively. Let R be the root system of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, and W the Weyl group of G. We consider a W-action $\bullet$ on Λ centered at $-\rho = -\frac{1}{2}\sum_{\alpha \in R^+} \alpha$:

$$w \bullet \lambda = w(\lambda + \rho) - \rho, \quad \lambda \in \Lambda.$$

If $\mathfrak{Z}_{\mathrm{HC}} = \mathbf{U}^{\mathrm{Ad}(G)} = \{u \in \mathbf{U} \mid \mathrm{Ad}(g)u = u \ \forall g \in G\}$ and $\mathfrak{h} = \mathrm{Lie}(T)$, transferring the $W\bullet$-action onto $\mathfrak{h}^*$, the Harish-Chandra isomorphism carries over:

$$\mathfrak{Z}_{\mathrm{HC}} \simeq \mathrm{S}(\mathfrak{H})^{W\bullet}.$$

Define a $\Bbbk$-algebra homomorphism

$$\begin{tikzcd} \mathfrak{Z}_{\mathrm{HC}} \arrow[r, dashed, "\mathrm{cen}_0"] \arrow[dr, phantom, "\circlearrowleft"] & \Bbbk & 0 \\ \mathrm{S}(\mathfrak{h})^{W\bullet} \arrow[u, "\sim"] \arrow[r, hook] & \mathrm{S}(\mathfrak{h}) \arrow[u] & h \in \mathfrak{h}, \arrow[u, mapsto] \end{tikzcd}$$

and set $\mathbf{U}^0 = \mathbf{U} \otimes_{\mathfrak{Z}_{\rm HC}} \mathrm{cen}_0$. Then the Beilinson-Bernstein localization theorem survives in the derived category:

Theorem [BMR] : *Assume $p > 2(h-1)$.*

(i) *The natural $\Bbbk$-algebra homomorphism $\mathbf{U} \to \Gamma(\mathcal{B}, \mathcal{D})$ induces an isomorphism*

$$\mathbf{U}^0 \to \Gamma(\mathcal{B}, \mathcal{D}).$$

(ii) *There is a derived equivalence between the category $\mathbf{U}^0\mathbf{mod}$ of $\mathbf{U}^0$-modules of finite type and the category $\mathrm{Coh}(\mathcal{D})$ of coherent $\mathcal{D}$-modules*

$$\mathrm{D}^b(\mathbf{U}^0\mathbf{mod}) \underset{\mathbb{R}\Gamma(\mathcal{B},?)}{\overset{\mathcal{D}\otimes^{\mathbb{L}}_{\mathbf{U}^0}?}{\rightleftarrows}} \mathrm{D}^b(\mathrm{Coh}(\mathcal{D})) .$$

(1.3) $\forall x \in \mathfrak{g}$, the p-th power x^p of x in $\mathrm{Dist}(G)$ lies in $\mathfrak{g}$, which we denote by $x^{[p]}$ to distinguish from the p-th power x^p in $\mathbf{U}$. Then

$$\mathfrak{Z}_{\rm Fr} = \Bbbk[x^p - x^{[p]} \mid x \in \mathfrak{g}]$$

is central in $\mathbf{U}$, called the Frobenius center of $\mathbf{U}$. If $x_1, \dots, x_r$ is a $\Bbbk$-linear basis of $\mathfrak{g}$, $\mathfrak{Z}_{\rm Fr}$ is the polynomial $\Bbbk$-algebra in $x_i^p - x_i^{[p]}$, and $\mathbf{U}$ is free over $\mathfrak{Z}_{\rm Fr}$ of basis x^n, $n \in [0, p[^r$:

$$\mathbf{U} = \coprod_{n \in [0,p[^r} \mathfrak{Z}_{\rm Fr}\, x^n.$$

Due to the large center of $\mathbf{U}$, any simple $\mathbf{U}$-module is of finite dimension [J98, 1.1].

By the standing hypothesis that $p > 2(h-1)$, the killing form κ on $\mathfrak{g}$ is nondegenerate. If $\mathcal{N} = \mathrm{Ad}(G)\mathfrak{n}$ the nilcone of $\mathfrak{g}$ and if $\mathrm{S}(\mathfrak{g})$ is the symmetric $\Bbbk$-algebra of $\mathfrak{g}$, one has $\Bbbk$-algebra homomorphisms

$$\begin{array}{ccccccc} \mathfrak{Z}_{\rm Fr} & \xleftarrow{\sim} & \mathrm{S}(\mathfrak{g})^{(1)} & \xrightarrow{\sim} & \Bbbk[\mathfrak{g}]^{(1)} & \xrightarrow{\mathrm{res}} & \Bbbk[\mathcal{N}]^{(1)} \\ x^p - x^{[p]} & \longleftarrow\!\!\shortmid & x & \longmapsto & \kappa(x,\ ?), & & x \in \mathfrak{g}, \end{array}$$

where $\mathrm{S}(\mathfrak{g})^{(1)}$ is the ring $\mathrm{S}(\mathfrak{g})$ with the $\Bbbk$-action twisted in such a way that each $\zeta \in \Bbbk$ act as $\zeta^{\frac{1}{p}}$ on $\mathrm{S}(\mathfrak{g})$, and likewise $\Bbbk[\mathfrak{g}]^{(1)}$, $\Bbbk[\mathcal{N}]^{(1)}$. Let $\forall \chi \in \mathcal{N}$, $\mathfrak{m}_\chi = \ker(\mathrm{ev}_{\chi^{(1)}} \circ \mathrm{res}) \in \mathrm{Max}(\mathfrak{Z}_{\rm Fr})$, $\mathbf{U}^0_\chi = \mathbf{U}^0 \otimes_{\mathfrak{Z}_{\rm Fr}} (\mathfrak{Z}_{\rm Fr}/\mathfrak{m}_\chi)$, and $\mathbf{U}^0\mathbf{mod}_\chi$ the full subcategory of $\mathbf{U}^0\mathbf{mod}$ consisting of those M such that $\mathfrak{m}_\chi^n M = 0$ $\exists n \in \mathbb{N}$, or equivalently, having support in the closed subscheme of $\mathrm{Spec}(\mathfrak{Z}_{\rm Fr})$ defined by $\mathfrak{m}_\chi$.

Likewise if $\mathrm{S}(\mathcal{T}_\mathcal{B})$ is the symmetric algebra of the tangent sheaf $\mathcal{T}_\mathcal{B}$ on $\mathcal{B}$,

$$\mathrm{Z}(\mathcal{D}) \simeq \mathrm{S}(\mathcal{T}_\mathcal{B})^{(1)} \quad \text{via} \quad a^p(\partial^p - \partial^{[p]}) \leftarrow\!\shortmid a^{(1)}\partial^{(1)}, a \in \mathcal{O}_\mathcal{B}, \partial \in \mathcal{T}_\mathcal{B} \simeq \mathcal{D}er_{\mathcal{B}/\Bbbk}.$$

If $q : \mathbb{V}(\mathcal{T}_\mathcal{B}) = \mathrm{Spec}(\mathrm{S}(\mathcal{T}_\mathcal{B})) \to \mathcal{B}$ is the cotangent bundle on $\mathcal{B}$, under the morphism

$$(1) \qquad \begin{array}{ccccccc} \mathbb{V}(\mathcal{T}_\mathcal{B}) & \xleftarrow{\sim} & G \times^B (\mathfrak{g}/\mathfrak{b})^* & \xrightarrow{\sim} & G \times^B \mathfrak{n} & \xrightarrow{p_2} & \mathcal{N} \\ & & & & [g,x] & \longmapsto & \mathrm{Ad}(g)x \end{array}$$

put $\mathcal{B}_\chi = \mathbb{V}(\mathcal{T}_\mathcal{B})\times_\mathcal{N}\chi$, called the Springer fiber of χ, $\mathcal{D}_\chi = \mathcal{D}\otimes_{\mathrm{Z}(\mathcal{D})}\{\mathrm{Z}(\mathcal{D})/p_2^\sharp(\mathrm{res}(\mathfrak{m}_\chi))\mathrm{Z}(\mathcal{D})\}$, and let $\mathrm{Coh}_\chi(\mathcal{D})$ be the full subcategory of $\mathrm{Coh}(\mathcal{D})$ consisting of those $\mathcal{M}$ with $p_2^\sharp(\mathrm{res}(\mathfrak{m}_\chi))^n\mathcal{M} = 0$ $\exists n \in \mathbb{N}$, or equivalently, such that $\mathrm{supp}(\tilde{q}^*\mathcal{M}) \subseteq (\mathcal{B}_\chi)^{(1)}$, where $\tilde{q} : (\mathbb{V}(\mathcal{T}_\mathcal{B})^{(1)}, \mathcal{O}_{\mathbb{V}(\mathcal{T}_\mathcal{B})^{(1)}}) \to (\mathcal{B}, \mathrm{Z}(\mathcal{D}))$ is the morphism of ringed spaces induced by q.

Theorem [BMR]: *Assume $p > 2(h-1)$.*

(i) *The BMR derived equivalence resricts to a derived equivalence*

$$\mathrm{D}^b(\mathbf{U}^0\mathbf{mod}_\chi) \simeq \mathrm{D}^b(\mathrm{Coh}_\chi(\mathcal{D})).$$

(ii) *There is a categorical equivalence*

$$\mathrm{Coh}(\mathcal{D}_\chi) \simeq \mathrm{Coh}(\mathcal{B}_\chi^{(1)}).$$

(iii) *If $\mathrm{K}(\mathcal{B}_\chi)$ is the Grothendieck group of $\mathrm{Coh}(\mathcal{B}_\chi)$ and if ℓ is a prime $\neq p$,*

$$\mathrm{rk}\,\mathrm{K}(\mathcal{B}_\chi) = \dim_{\bar{\mathbb{Q}}_\ell} \mathrm{H}^\bullet_{\mathrm{et}}(\mathcal{B}_\chi, \bar{\mathbb{Q}}_\ell).$$

(1.5) **Corollary [BMR]:** *The number of irreducibles for $\mathbf{U}^0_\chi$ is equal to*

$$\dim_{\bar{\mathbb{Q}}_\ell} \mathrm{H}^\bullet_{\mathrm{et}}(\mathcal{B}_\chi, \bar{\mathbb{Q}}_\ell).$$

(1.6) We wish to make the BMR-theory T-equivariant to keep track of the weights. In order for T to act on $\mathbf{U}_\chi = \mathbf{U}/(\mathfrak{m}_\chi)$ by Ad,

$$(\mathfrak{m}_\chi) = \mathbf{U}\mathfrak{m}_\chi = (x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}) \triangleleft \mathbf{U}$$

must be $\mathrm{Ad}(T)$-invariant, which forces $\chi = 0$. Thus in the T-equivariant theory we are to deal with $\mathbf{U}_0 \simeq \mathrm{Dist}(G_1)$, $G_1 = \ker(Fr : G \to G^{(1)})$ the Frobenius kernel of G, and the BMR derived equivalence reads

$$\mathrm{D}^b(\mathbf{U}^0\mathbf{mod}_0) \underset{\mathbb{R}\Gamma(\mathcal{B},?)}{\overset{\mathcal{D}\otimes^{\mathbb{L}}_{\mathbf{U}^0}?}{\rightleftarrows}} \mathrm{D}^b(\mathrm{Coh}_0(\mathcal{D}))\ .$$

2° Arithmetic differential operators

(2.1) Let X be a smooth $\Bbbk$-variety. The sheaf $\mathcal{D}_X$ of $\Bbbk$-algebras of crystalline differential operators on X coincides with the 0-th term $\mathcal{D}_X^{(0)}$ of Berthelot's sheaves $\mathcal{D}_X^{(m)}$, $m \in \mathbb{N}$, of $\Bbbk$-algebras of arithmetic differential operators on X [B96]. The $\mathcal{D}_X^{(m)}$ form an inductive system such that for $m' \geq m$

$$\begin{tikzcd}
\mathcal{D}_X^{(m')} \arrow[r, "\rho_{m'}"] & \mathcal{D}iff_X \\
\mathcal{D}_X^{(m)} \arrow[u, "\rho_{m',m}"] \arrow[r, two heads, "\rho_m"'] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{M}od_{\mathcal{O}_\mathcal{B}^{[m+1]}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B}), \arrow[u, hook]
\end{tikzcd}$$

where $\mathcal{O}_X^{[m+1]} = \{a^{p^{m+1}} \mid a \in \mathcal{O}_X\}$; $(\mathcal{M}od_{\mathcal{O}_\mathcal{B}^{[m]}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B}) \mid m \in \mathbb{N})$ forms the p-filtration of $\mathcal{D}iff_X$ studied by Haastert [H87, 88]. It will follow from the structural information (2.2) below that

$$\varinjlim_m \mathcal{D}_X^{(m)} \simeq \mathcal{D}iff_X,$$

and we will write $\mathcal{D}_X^{(\infty)}$ for $\mathcal{D}iff_X$; $\mathcal{D}_X^{(0)}$ can be defined in characteristic 0 and is isomorphic to $\mathcal{D}iff_X$ there. Put $\mathcal{K}_m = \ker(\rho_m)$.

(2.2) Let $(t_1, \dots, t_d)$ be a local coordinate on an open U of X. Recall from [EGAIV] that $\mathcal{D}_U^{(\infty)} = \mathcal{D}iff_U$ is free over $\mathcal{O}_U$ of basis $\partial^{[n]}$, $n \in \mathbb{N}^d$, such that

$$\partial^{[n]}(t^k) = \binom{k}{n} t^{k-n} \qquad \forall k \in \mathbb{N}^d.$$

Proposition [B96, 2.2.3-7]: *Let $m \in \mathbb{N}$.*

(i) $\mathcal{D}_U^{(m)}$ *is free over* $\mathcal{O}_U$ *of basis* $\partial^{<n>}$, $n \in \mathbb{N}^d$, *such that* $\forall k, n' \in \mathbb{N}^d$, $\forall a \in \mathcal{O}_U$,

$$\begin{aligned}
\rho_m(\partial^{<n>}) &= q!\partial^{[n]}, \\
\partial^{<n>}(t^k) := \rho_m(\partial^{<n>})(t^k) &= q!\binom{k}{n} t^{k-n}, \\
\partial^{<n>}\partial^{<n'>} &= \left\langle \begin{matrix} n+n' \\ n \end{matrix} \right\rangle \partial^{<n+n'>}, \\
\partial^{<n>} a &= \sum_{n'+n''=n} \left\{ \begin{matrix} n \\ n' \end{matrix} \right\} \partial^{<n'>}(a)\partial^{<n''>},
\end{aligned}$$

where $q = (q_i) \in \mathbb{N}^d$ *with* $n_i = p^m q_i + r_i$, $r_i \in [0, p^m[$ $\forall i \in [1, d]$,

$$\left\{ \begin{matrix} n \\ n' \end{matrix} \right\} = \frac{q!}{q'!q''!} \quad \text{with } q' \text{ and } q'' \text{ defined for } n' \text{ and } n'', \text{ resp., as } q \text{ for } n,$$

$$\left\langle \begin{matrix} n+n' \\ n \end{matrix} \right\rangle = \binom{n+n'}{n} \left\{ \begin{matrix} n+n' \\ n \end{matrix} \right\}^{-1}.$$

Thus $\mathcal{D}_U^{(m)} = \mathcal{O}_U[\partial_i^{<p^j>} \mid i \in [1, d], j \in [0, m]]$, *and hence is left and right noetherian.*

(ii) *The center* $\mathrm{Z}(\mathcal{D}_U^{(m)})$ *of* $\mathcal{D}_U^{(m)}$ *is a polynomial* $\mathcal{O}_U^{[m+1]}$*-algebra in indeterminates* $\partial_i^{<p^{m+1}>}$, $i \in [1, d]$.

(iii) *If* $m' > m$, $\rho_{m',m}(\partial^{<n>}) = \dfrac{q!}{q'!}\partial^{<n>}$ *with* $q' \in \mathbb{N}^d$ *defined by* $n_i = p^{m'} q'_i + r'_i$, $r'_i \in [0, p^{m'}[$ $\forall i \in [1, d]$, *and*

$$\ker(\rho_{m',m}|_U) = (\partial_i^{<p^{m+1}>} \mid i \in [1, d]) = \mathcal{K}_m|_U.$$

(2.3) It is now easy to generalize a result of [BMR] that $\mathcal{D}_X^{(0)}$ is Azumaya:

Theorem: *Each $\mathcal{D}_X^{(m)}$, $m \in \mathbb{N}$, is Azumaya; if $\mathcal{A}_X = \mathcal{O}_X[\mathrm{Z}(\mathcal{D}_X^{(m)})]$, there is an isomorphism of sheaves of $\Bbbk$-algebras on X*

$$\mathcal{D}_X^{(m)} \otimes_{\mathrm{Z}(\mathcal{D}_X^{(m)})} \mathcal{A}_X \simeq \mathcal{M}od(\mathcal{A}_X)(\mathcal{D}_X^{(m)}, \mathcal{D}_X^{(m)}) \quad \textit{via} \quad \delta \otimes \delta' \mapsto \delta ? \delta',$$

where the RHS is the sheaf of endomorphisms of right $\mathcal{A}_X$-module $\mathcal{D}_X^{(m)}$.

Proof: By [KO, III.6.6, p.104] the question being local, we may assume X is affine with coordinate system $(t_1, \dots, t_d)$. Put $D = \Gamma(X, \mathcal{D}_X^{(m)})$, $Z = \Gamma(X, Z(\mathcal{D}_X^{(m)}))$ and $A = \Gamma(X, \mathcal{A}_X)$. Then

(1) $$A = \coprod_{k \in [0, p^{m+1}[^d} Z t^k,$$

(2) $$D = \coprod_{k \in [0, p^{m+1}[^d} A \partial^{<k>} = \coprod_{k \in [0, p^{m+1}[^d} \partial^{<k>} A \quad \text{by (2.2.i)/[B96, 2.2.5.1]}$$
$$= \coprod_{k, n \in [0, p^{m+1}[^d} Z t^k \partial^{<n>}.$$

We have thus only to show

(3) $$D \otimes_Z A \simeq \mathbf{Mod}A(D, D) \quad \text{via} \quad \delta \otimes \delta' \mapsto \delta ? \delta'.$$

For that, both sides being free over A of the same rank, it is enough by NAK [AM, 2.7+3.9] to verify the surjectivity of (3) at each maximal ideal of A: $\forall \mathfrak{m} \in \mathrm{Max}(A)$,

$$\begin{array}{ccc} D \otimes_Z A \otimes_A A(\mathfrak{m}) & \twoheadrightarrow & \mathbf{Mod}A(D, D) \otimes_A A(\mathfrak{m}) \\ \sim \uparrow & & \downarrow \sim \\ D \otimes_Z A(\mathfrak{m}) & & \mathbf{Mod}A(\mathfrak{m})(D \otimes_A A(\mathfrak{m}), D \otimes_A A(\mathfrak{m})). \end{array}$$

The surjectivity, in turn, will follow by Jacobson's density theorem [L, p.647] from the irreducibility of $D \otimes_A A(\mathfrak{m})$ as left $D \otimes_Z A(\mathfrak{m})$-module.

Put $B = \Bbbk[X]$. As $A = B[Z]$ is the polynomial B-algebra in indeterminates $\partial_1^{<p^{m+1}>}, \dots, \partial_d^{<p^{m+1}>}$ by (2.2.ii),

$$\mathrm{Max}(A) \simeq \mathbb{A}_B^d \simeq \mathrm{Max}(B) \times \mathbb{A}_\Bbbk^d.$$

At $(x, y) \in \mathrm{Max}(B) \times \mathbb{A}_\Bbbk^d$,

$$D \otimes_A A(\mathfrak{m}) = \coprod_{k \in [0, p^{m+1}[^d} \Bbbk \partial^{<k>}, \quad D \otimes_Z A(\mathfrak{m}) = \coprod_{k, n \in [0, p^{m+1}[^d} \Bbbk t^k \partial^{<n>}.$$

We may assume $t_i(x) = 0$ $\forall i$. By (2.2.i)/[B96, 2.2.5.1] again we have only to show

(4) $$(D \otimes_Z A(\mathfrak{m}))\delta \ni 1 \quad \forall \delta \in \coprod_{k \in [0, p^{m+1}[^d} \Bbbk \partial^{<k>} \setminus 0.$$

Applying the adjoint operator on the 4-th formula in (2.2.i) yields

$$(-1)^{|k|} b \partial^{<k>} = \sum_{k' + k'' = k} \left\{ \begin{matrix} k \\ k' \end{matrix} \right\} (-1)^{|k''|} \partial^{<k''>} \partial^{<k'>}(b) \quad \forall k \in \mathbb{N}^d \ \forall b \in B,$$

where $|k| = \sum_{i=1}^{d} k_i$ and likewise $|k''|$. Consequently, if $k_i \geq 1$, one has in $D \otimes_Z A(\mathfrak{m})$

$$\begin{aligned}
(-1)^{|k|} t_i \partial^{<k>} &= \sum_{k' \neq 0} \left\{ \begin{matrix} k \\ k' \end{matrix} \right\} (-1)^{|k-k'|} \partial^{<k-k'>} \partial^{<k'>}(t_i) \\
&= \left\{ \begin{matrix} k \\ 1_i \end{matrix} \right\} (-1)^{|k-1_i|} \partial^{<k-1_i>} \quad \text{with } 1_i \in \mathbb{N}^d \text{ such that } 1_{ij} = \delta_{ij} \ \forall j \in [1,d] \\
&\in \mathbb{k}^\times \partial^{<k-1_i>} \quad \text{as } q_{kj} \leq p-1 \ \forall j \in [1,d],
\end{aligned}$$

and (4) will follow.

Remark: *As in [BMR] one has* $\mathcal{A}_X = \mathrm{C}_{\mathcal{D}_X^{(m)}}(\mathcal{O}_X)$.

(2.4) **Inverse image:** In order to treat $\mathcal{D}_X^{(m)}$, $m \in \mathbb{N}$, and $\mathcal{D}_X^{(\infty)} = \mathcal{D}iff_X$ simultaneously, put $\bar{\mathbb{N}} = \mathbb{N} \sqcup \{\infty\}$. Let $f : X \to Y$ be a morphism of smooth $\mathbb{k}$-varieties. Denote the category of quasi-coherent left $\mathcal{D}_X^{(m)}$- (resp. $\mathcal{D}_Y^{(m)}$-) modules by $\mathrm{qc}(\mathcal{D}_X^{(m)})$ (resp. $\mathrm{qc}(\mathcal{D}_Y^{(m)})$), $m \in \bar{\mathbb{N}}$.

If $\mathcal{V} \in \mathrm{qc}(\mathcal{D}_Y^{(m)})$, $f^*(\mathcal{V}) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{V}$ comes equipped with a structure of quasi-coherent left $\mathcal{D}_X^{(m)}$-module [B00, 2.1.1] such that, suppressing (m), locally

$$\begin{aligned}
\partial_X^{<k>} \cdot (1 \otimes v) &= \sum_{|j| \leq |k|} \partial_X^{<k>}((f \times f)^\sharp(\tau_Y^{\{j\}})) \otimes \partial_Y^{<j>} v \\
&\qquad \text{by Taylor's expansion formula [B96, 2.3.2.2]} \\
&= \sum_j \partial_X^{<k>}((f \times f)^\sharp(\tau_Y))^{\{j\}} \otimes \partial_Y^{<j>} v \\
&\qquad \text{as } (f \times f)^\sharp \text{ is an } m\text{-PD-morphism by [B96, 2.1.4]},
\end{aligned}$$

where $\tau_Y = \tau_{Y,1} \dots \tau_{Y,d_Y}$, $\tau_{Y,i} = 1 \otimes t_{Y,i} - t_{Y,i} \otimes 1$ in the sheaf $\mathcal{P}^{|k|}_{Y/\mathbb{k},(m)}$ of the principal parts of level m and of order $|k|$ of Y over $\mathbb{k}$, if $(t_{Y,1}, \dots, t_{Y,d_Y})$ is a local coordinate on Y, and

$$(f \times f)^\sharp(\tau_Y)^{\{j\}} = (f \times f)^\sharp(\tau_Y)^r \gamma_q((f \times f)^\sharp(\tau_Y)^{p^m})$$

if $j = p^m q + r$ with γ the PD-structure on $\mathcal{P}^{|k|}_{Y/\mathbb{k},(m)}$ [B96, 1.3.5.1]. One thus obtains a functor $\forall m \in \bar{\mathbb{N}}$

$$f^* : \mathrm{qc}(\mathcal{D}_Y^{(m)}) \to \mathrm{qc}(\mathcal{D}_X^{(m)}).$$

In particular, $f^*(\mathcal{D}_Y^{(m)})$ carries a structure of $(\mathcal{D}_X^{(m)}, f^{-1}\mathcal{D}_Y^{(m)})$-bimodule, denoted $\mathcal{D}_{f\to}^{(m)}$. Then

$$f^* \simeq \mathcal{D}_{f\to}^{(m)} \otimes_{f^{-1}(\mathcal{D}_Y^{(m)})} f^{-1}(?).$$

If $m' \in [m, \infty]$, the morphism $f^*(\rho_{m',m}) : \mathcal{D}_{f\to}^{(m)} \to \mathcal{D}_{f\to}^{(m')}$ is compatible with the structure of $(\mathcal{D}_X^{(m)}, f^{-1}\mathcal{D}_Y^{(m)})$-, $(\mathcal{D}_X^{(m')}, f^{-1}\mathcal{D}_Y^{(m')})$-bimodules:

(1)
$$\begin{tikzcd}
\mathcal{D}_X^{(m)} \times \mathcal{D}_{f\to}^{(m)} \times f^{-1}\mathcal{D}_X^{(m)} \arrow[r] \arrow[d, "\rho_{m',m} \times f^*(\rho_{m',m}) \times f^{-1}(\rho_{m',m})"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}_{f\to}^{(m)} \arrow[d, "f^*(\rho_{m',m})"] \\
\mathcal{D}_X^{(m')} \times \mathcal{D}_{f\to}^{(m')} \times f^{-1}\mathcal{D}_X^{(m')} \arrow[r] & \mathcal{D}_{f\to}^{(m')} .
\end{tikzcd}$$

If $g : Y \to Z$ is another morphism of smooth $\Bbbk$-varieties, from [B00, 2.1.1]

$$(g \circ f)^* \simeq f^* \circ g^*.$$

(2.5) **Direct image:** Keep the notations of (2.4). $\forall m \in \bar{\mathbb{N}}$, denote the category of quasi-coherent right $\mathcal{D}_X^{(m)}$- (resp. $\mathcal{D}_Y^{(m)}$-) modules by $\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X^{(m)})$ (resp. $\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y^{(m)})$). We define the direct image functor $f^{\mathrm{rgt}}_{+,(m)} : \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X^{(m)}) \to \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y^{(m)})$ for right modules by

$$f^{\mathrm{rgt}}_{+,(m)} = f_*(? \otimes_{\mathcal{D}_X} \mathcal{D}^{(m)}_{f\to}),$$

using the structure of right $f^{-1}\mathcal{D}_Y^{(m)}$-module on $\mathcal{D}^{(m)}_{f\to}$ [B00, 2.1.3] as in [H88, 3.1]. If ω_X is the dualizing sheaf on X, ω_X is equipped with a structure of right $\mathcal{D}_X^{(\infty)}$-module, and hence of right $\mathcal{D}_X^{(m)}$-module for each m via ρ_m, and defines an equivalence of categories [B00, 1.2.7]

$$\mathrm{qc}(\mathcal{D}_X^{(m)}) \underset{? \otimes_X \omega_X^{-1}}{\overset{\omega_X \otimes_X ?}{\rightleftarrows}} \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X^{(m)}).$$

Then we define the direct image functor $\int^0_{f,(m)} : \mathrm{qc}(\mathcal{D}_X^{(m)}) \to \mathrm{qc}(\mathcal{D}_Y^{(m)})$, as in [H88, 7.1], to be

$$\int^0_{f,(m)} = (? \otimes_Y \omega_Y^{-1}) \circ f^{\mathrm{rgt}}_{+,(m)} \circ (\omega_X \otimes_X ?).$$

Alternatively, $f^*(\mathcal{D}_Y \otimes_Y \omega_Y^{-1})$ is equipped with two isomorphic natural structures of left $(f^{-1}\mathcal{D}_Y^{(m)}, \mathcal{D}_X^{(m)})$-modules [B00, 3.4.1], and defines a $(f^{-1}\mathcal{D}_Y^{(m)}, \mathcal{D}_X^{(m)})$-bimodule $\mathcal{D}^{(m)}_{f\leftarrow} = \omega_X \otimes_X f^*(\mathcal{D}_Y^{(m)} \otimes_Y \omega_Y^{-1})$. One has as in [H88, 7.1]

$$\int^0_{f,(m)} \simeq f_*(\mathcal{D}^{(m)}_{f\leftarrow} \otimes_{\mathcal{D}_X^{(m)}} ?).$$

In case f is an open immersion,

$$\int^0_{f,(m)} \simeq f_*.$$

If $m' \in [m, \infty]$, the morphism $\omega_X \otimes_X f^*(\rho_{m',m} \otimes_Y \omega_Y^{-1}) : \mathcal{D}^{(m)}_{f\leftarrow} \to \mathcal{D}^{(m')}_{f\leftarrow}$ is compatible with the structure of $(f^{-1}\mathcal{D}_Y^{(m)}, \mathcal{D}_X^{(m)})$-, $(f^{-1}\mathcal{D}_Y^{(m')}, \mathcal{D}_X^{(m')})$-bimodules:

(1)
$$\begin{tikzcd}
f^{-1}\mathcal{D}_Y^{(m)} \times \mathcal{D}^{(m)}_{f\leftarrow} \times \mathcal{D}_X^{(m)} \arrow[r] \arrow[d, "f^{-1}(\rho_{m',m}) \times (\omega_X \otimes_X f^*(\rho_{m',m} \otimes_Y \omega_Y^{-1})) \times \rho_{m',m}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}^{(m)}_{f\leftarrow} \arrow[d, "\omega_X \otimes_X f^*(\rho_{m',m} \otimes_Y \omega_Y^{-1})"] \\
f^{-1}\mathcal{D}_Y^{(m')} \times \mathcal{D}^{(m')}_{f\leftarrow} \times \mathcal{D}_X^{(m')} \arrow[r] & \mathcal{D}^{(m')}_{f\leftarrow}.
\end{tikzcd}$$

If $g : Y \to Z$ is another morphism of smooth $\Bbbk$-varieties,

$$\int^0_{g\circ f,(m)} \simeq \int^0_{g,(m)} \circ \int^0_{f,(m)}.$$

In the derived category we set

$$\int_{f,(m)} = \mathbb{R}f_*(\mathcal{D}^{(m)}_{f\leftarrow}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} ?) : \mathrm{D}^b(\mathrm{qc}(\mathcal{D}^{(m)}_X)) \to \mathrm{D}^b(\mathrm{qc}(\mathcal{D}^{(m)}_Y)).$$

(2.6) $\forall m \in \mathbb{N}$, put $\bar{\mathcal{D}}^{(m)}_X = \mathrm{im}(\rho_m) = \mathcal{M}od_{\mathcal{O}^{[m+1]}_X}(\mathcal{O}_X, \mathcal{O}_X)$. Haastert [H88] denoted $\bar{\mathcal{D}}^{(m)}_X$ by $\mathcal{D}_{X,m+1}$, and defined the direct image functor with respect to $\bar{\mathcal{D}}^{(m)}_X$ and $\bar{\mathcal{D}}^{(m)}_Y$ for each $m \in \mathbb{N}$ by

$$\mathcal{M} \mapsto f_*(\bar{\mathcal{D}}^{(m)}_{f\leftarrow} \otimes_{\bar{\mathcal{D}}^{(m)}_X} \mathcal{M}) \quad \text{with} \quad \bar{\mathcal{D}}^{(m)}_{f\leftarrow} = \omega_X \otimes_X f^*(\bar{\mathcal{D}}^{(m)}_Y \otimes_Y \omega_Y^{-1}),$$

which we will denote by $\bar{\int}^0_{f,(m)} : \mathrm{qc}(\bar{\mathcal{D}}^{(m)}_X) \to \mathrm{qc}(\bar{\mathcal{D}}^{(m)}_Y)$, denoted in [H88] by $\displaystyle\int^0_{f,m+1}$. There is an isomorphism of $(f^{-1}\mathcal{D}^{(\infty)}_Y, \mathcal{D}^{(\infty)}_X)$-bimodules

$$\mathcal{D}^{(\infty)}_{f\leftarrow} \simeq \varinjlim_m \bar{\mathcal{D}}^{(m)}_{f\leftarrow},$$

to yield

$$\int^0_{f,(\infty)} \simeq \varinjlim_m \bar{\int}^0_{f,(m)} : \mathrm{qc}(\mathcal{D}^{(\infty)}_X) \to \mathrm{qc}(\mathcal{D}^{(\infty)}_Y).$$

$\forall m \in \mathbb{N}$, $\bar{\mathcal{D}}^{(m)}_{f\leftarrow}$ is locally free as right $\bar{\mathcal{D}}^{(m)}_X$-module [H88, 1.2], and hence $\bar{\mathcal{D}}^{(\infty)}_{f\leftarrow}$ is flat over $\bar{\mathcal{D}}^{(\infty)}_X$. It follows that all $\bar{\int}^0_{f,(m)}$ and $\displaystyle\int^0_{f,(\infty)}$ are left exact. Put for simplicity

$$\int^0_f = \int^0_{f,(\infty)}.$$

To relate $\displaystyle\int^0_{f,(m)}$ to $\bar{\int}^0_{f,(m)}$, we have

Proposition: $\forall m \in \bar{\mathbb{N}}$,

$$\bar{\mathcal{D}}^{(m)}_Y \otimes_{\mathcal{D}^{(m)}_Y} \int^0_{f,(m)} \simeq \bar{\int}^0_{f,(m)} : \mathrm{qc}(\bar{\mathcal{D}}^{(m)}_X) \to \mathrm{qc}(\bar{\mathcal{D}}^{(m)}_Y).$$

In particular, $\displaystyle\varinjlim_m \int^0_{f,(m)} \simeq \int^0_{f,(\infty)} = \int^0_f$ *on* $\mathrm{qc}(\mathcal{D}^{(\infty)}_X)$.

Proof: Consider a natural morphism

(1)

$$\begin{tikzcd}
\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} \int^0_{f,(m)} \mathcal{M} \arrow[r, dashed] \arrow[d, equal] & \bar{\int}^0_{f,(m)} \mathcal{M} \arrow[d, equal] \\
\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} f_*(\mathcal{D}^{(m)}_{f\leftarrow} \otimes_{\mathcal{D}_X^{(m)}} \mathcal{M}) \arrow[d, equal] \arrow[r, phantom, "\circlearrowleft"] & f_*(\bar{\mathcal{D}}^{(m)}_{f\leftarrow} \otimes_{\bar{\mathcal{D}}_X^{(m)}} \mathcal{M}) \arrow[d, equal] \\
\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} f_*\{(\omega_X \otimes_X f^*_{\mathrm{lft}}(\mathcal{D}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\mathcal{D}_X^{(m)}} \mathcal{M}\} \arrow[d, "\sim"'] & f_*\{(\omega_X \otimes_X f^*_{\mathrm{lft}}(\bar{\mathcal{D}}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\bar{\mathcal{D}}_X^{(m)}} \mathcal{M}\} \arrow[d, "\sim"] \\
\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} f_*\{(\omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\mathcal{D}_X^{(m)}} \mathcal{M}\} \arrow[r] & f_*\{(\omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\bar{\mathcal{D}}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\bar{\mathcal{D}}_X^{(m)}} \mathcal{M}\} \\
\bar{\delta}_1 \otimes a \otimes \delta_2 \otimes b \otimes m \arrow[r, mapsto] & \bar{\delta}_1 \cdot (a \otimes \bar{\delta}_2 \otimes b \otimes m),
\end{tikzcd}$$

which is well-defined by (2.5.1). To see it invertible, the question being local, we may assume Y is affine. Using an affine open cover, we may also assume X is affine. Then (1) reads as

$$\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} f_*\{(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\mathcal{D}_X^{(m)}} \mathcal{M}\} \to \\ f_*\{(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\bar{\mathcal{D}}_Y^{(m)} \otimes_Y \omega_Y^{-1})) \otimes_{\bar{\mathcal{D}}_X^{(m)}} \mathcal{M}\}$$

via

$$\bar{\delta}_1 \otimes a \otimes \delta_2 \otimes m \mapsto \bar{\delta}_1 \cdot (a \otimes \bar{\delta}_2 \otimes m) = a \otimes \bar{\delta}_2({}^t\bar{\delta}_1) \otimes m$$

with inverse ${}^t\bar{\delta}_2 \otimes a \otimes 1 \otimes m \longleftarrow a \otimes \bar{\delta}_2 \otimes m$.

It follows in the limit that

$$\begin{aligned}
\int_{f,(\infty)} &\simeq \varinjlim_m \bar{\int}_{f,(m)} \quad \text{by [H88, 7.1]} \\
&\simeq \varinjlim_m \{\bar{\mathcal{D}}_Y^{(m)} \otimes_{\mathcal{D}_Y^{(m)}} \int_{f,(m)}\} \\
&\simeq (\varinjlim_m \bar{\mathcal{D}}_Y^{(m)}) \otimes_{(\varinjlim_m \mathcal{D}_Y^{(m)})} (\varinjlim_m \int_{f,(m)}) \quad \text{by [BA, II.6.7 Prop.12]} \\
&\simeq \mathcal{D}_Y^{(\infty)} \otimes_{\mathcal{D}_Y^{(\infty)}} (\varinjlim_m \int_{f,(m)}) \\
&\simeq \varinjlim_m \int_{f,(m)}.
\end{aligned}$$

(2.7) **Kashiwara's equivalence [Kas70]:** $\forall m \in \bar{\mathbb{N}}$, after the functor

$$\bar{f}^+_{\mathrm{rgt},(m)} = \mathcal{M}od(f^{-1}\bar{\mathcal{D}}_Y^{(m)})(\bar{\mathcal{D}}^{(m)}_{f\to}, f^{-1}\,?) : \mathrm{qc}^{\mathrm{rgt}}(\bar{\mathcal{D}}_Y^{(m)}) \to \mathrm{qc}^{\mathrm{rgt}}(\bar{\mathcal{D}}_X^{(m)})$$

in [H88], define a functor

$$f^+_{\mathrm{rgt},(m)} = \mathcal{M}od(f^{-1}\mathcal{D}_Y^{(m)})(\mathcal{D}^{(m)}_{f\to}, f^{-1}\,?) : \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y^{(m)}) \to \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X^{(m)}).$$

As in [H88, 8.12]:

$$(? \otimes_X \omega_X^{-1}) \circ \bar{f}^+_{\mathrm{rgt},(m)} \circ (\omega_Y \otimes_Y ?) \simeq (f^{-1}\bar{\mathcal{D}}_Y^{(m)})\mathcal{M}od(\bar{\mathcal{D}}^{(m)}_{f\leftarrow}, f^{-1}\,?) : \mathrm{qc}(\bar{\mathcal{D}}_Y^{(m)}) \to \mathrm{qc}(\bar{\mathcal{D}}_X^{(m)}),$$

which we denote by $\bar{f}^+_{(m)}$, one obtains

$$(?\otimes_X \omega_X^{-1})\circ f^+_{\mathrm{rgt},(m)}\circ(\omega_Y\otimes_Y ?)\simeq (f^{-1}\mathcal{D}_Y^{(m)})\mathcal{M}od(\mathcal{D}^{(m)}_{f\leftarrow}, f^{-1}\,?) : \mathrm{qc}(\mathcal{D}_Y^{(m)})\to \mathrm{qc}(\mathcal{D}_X^{(m)}),$$

which we will denote by $f^+_{(m)}$.

Assume in the rest of §2 that f is a closed immersion defined by an ideal sheaf $\mathcal{I}_X$ of $\mathcal{O}_Y$. $\forall m\in\bar{\mathbb{N}}$, let $\mathrm{qc}^{\mathrm{rgt}}_X(\bar{\mathcal{D}}_Y^{(m)})$ be the full subcategory of $\mathrm{qc}^{\mathrm{rgt}}(\bar{\mathcal{D}}_Y^{(m)})$ consisting of those $\mathcal{M}$ with $\mathrm{supp}(\mathcal{M})\subseteq X$. $\forall m\in\mathbb{N}$, let $\mathcal{I}_X^{[m]}=\{a^{p^m}\mid a\in\mathcal{I}_X\}$ and let $\mathrm{qc}^{\mathrm{rgt}}_{[X^{(m+1)}]}(\bar{\mathcal{D}}_Y^{(m)})$ be the full subcategory of $\mathrm{qc}^{\mathrm{rgt}}(\bar{\mathcal{D}}_Y^{(m)})$ consisting of those $\mathcal{M}$ annihilated by $\mathcal{I}_X^{[m+1]}$. Define likewise $\mathrm{qc}_{[X^{(m+1)}]}(\bar{\mathcal{D}}_Y^{(m)})$ and $\mathrm{qc}_X(\bar{\mathcal{D}}_Y^{(m)})$ for left modules.

As f is a closed immersion, all $\bar{\int}^0_{f,(m)}$, $\displaystyle\int^0_{f,(m)}$, $m\in\bar{\mathbb{N}}$, are exact, so that we may suppress 0 from those.

Theorem [H88]: (i) *$\forall m\in\mathbb{N}$, $\bar{f}^+_{\mathrm{rgt},(m)}\mid_{\mathrm{qc}_X(\bar{\mathcal{D}}_Y^{(m)})}$ is right adjoint to $\bar{f}^{\mathrm{rgt}}_{+,(m)}$, and hence taking direct limit, $f^+_{\mathrm{rgt},(\infty)}\mid_{\mathrm{qc}_X(\mathcal{D}_Y^{(\infty)})}$ is right adjoint to $f^{\mathrm{rgt}}_{+,(\infty)}$.*

(ii) *$\forall m\in\mathbb{N}$, $\bar{f}^+_{(m)}\mid_{\mathrm{qc}_X(\bar{\mathcal{D}}_Y^{(m)})}$ is right adjoint to $\bar{\int}_{f,(m)}$, and hence taking direct limit, $f^+_{(\infty)}\mid_{\mathrm{qc}_X(\mathcal{D}_Y^{(\infty)})}$ is right adjoint to $\displaystyle\int_f=\int_{f,(\infty)}$.*

(iii) *There are categorical equivalences*

$$\mathrm{qc}(\bar{\mathcal{D}}_X^{(m)}) \underset{\bar{f}^+_{(m)}}{\overset{\bar{\int}_{f,(m)}}{\rightleftarrows}} \mathrm{qc}_{[X^{(m+1)}]}(\bar{\mathcal{D}}_Y^{(m)}) \qquad \forall m\in\mathbb{N},$$

and hence also

$$\mathrm{qc}(\mathcal{D}_X^{(\infty)}) \underset{f^+_{(\infty)}}{\overset{\int_f}{\rightleftarrows}} \mathrm{qc}_X(\mathcal{D}_Y^{(\infty)}).$$

(2.8) In the limit $\displaystyle\varinjlim_m\int_{f,(m)}\simeq\int_f$ Kashiwara's equivalence holds by (2.7). At each $m\in\mathbb{N}$, however, $\displaystyle\int_{f,(m)}$ fails to induce an equivalence: define the subcategories $\mathrm{qc}^{\mathrm{rgt}}_X(\mathcal{D}_Y^{(m)})$ of $\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y^{(m)})$, and $\mathrm{qc}_X(\mathcal{D}_Y^{(m)})$ of $\mathrm{qc}(\mathcal{D}_Y^{(m)})$ just as for $\bar{\mathcal{D}}_Y^{(m)}$.

Proposition: *Let $m\in\mathbb{N}$.*

(i) *Each $f^+_{\mathrm{rgt},(m)}\mid_{\mathrm{qc}^{\mathrm{rgt}}_X(\mathcal{D}_Y^{(m)})}$ is right adjoint to $f^{\mathrm{rgt}}_{+,(m)}$; hence also each $f^+_{(m)}\mid_{\mathrm{qc}_X(\mathcal{D}_Y^{(m)})}$ is right adjoint to $\displaystyle\int_{f,(m)}$.*

(ii) $\forall \mathcal{L} \in \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X^{(m)}) \setminus 0$, *unless f is invertible, the adjunction*

$$\mathcal{L} \to f^{+}_{\mathrm{rgt},(m)} \circ f^{\mathrm{rgt}}_{+,(m)}(\mathcal{L})$$

is not epic; hence also the adjunction

$$\begin{aligned}
\mathcal{L} \otimes_X \omega_X^{-1} \to (f^{+}_{\mathrm{rgt},(m)} &\circ f^{\mathrm{rgt}}_{+,(m)})(\mathcal{L}) \otimes_X \omega_X^{-1} \\
&= \{(? \otimes_X \omega_X^{-1}) \circ f^{+}_{\mathrm{rgt},(m)} \circ (\omega_Y \otimes_Y ?) \circ (? \otimes_Y \omega_Y^{-1}) \circ f^{\mathrm{rgt}}_{+,(m)} \circ (\omega_X \otimes_X ?)\}(\mathcal{L} \otimes_X \omega_X^{-1}) \\
&= f^{+}_{(m)} \circ \int_{f,(m)} (\mathcal{L} \otimes_X \omega_X^{-1})
\end{aligned}$$

is not epic.

Proof: The arguments are the same as in [K?]. To illustrate, consider for example the case $Y = \mathrm{Spec}(\Bbbk[x, y])$, $X = \mathrm{Spec}(\Bbbk[x, y]/(y))$. Put $A = \Bbbk[x, y]$, $\bar{A} = \Bbbk[x] \simeq \Bbbk[x, y]/(y)$, $D^{(m)}(A) = \Gamma(Y, \mathcal{D}_Y^{(m)}) = \coprod_{i,j\in\mathbb{N}} A\partial_x^{<i>}\partial_y^{<y>}$, $D^{(m)}(\bar{A}) = \Gamma(X, \mathcal{D}_X^{(m)}) = \coprod_{i\in\mathbb{N}} \bar{A}\partial_x^{<i>}$, and $D^{(m)}_{f\to} = \Gamma(X, \mathcal{D}^{(m)}_{f\to})$.

If L is a $D^{(m)}(\bar{A})$-module, the adjunction reads as

$$\begin{tikzcd}
\ell \arrow[rr, mapsto] & & \ell \otimes ? \\
L \arrow[rr] \arrow[ddddrr, dashed] & & \mathbf{Mod}D^{(m)}(A)(D^{(m)}_{f\to}, L \otimes_{D^{(m)}(\bar{A})} D^{(m)}_{f\to}) \arrow[d, dash, "\sim"] \\
\ell \arrow[dddr, mapsto] & & \mathbf{Mod}D^{(m)}(A)(\bar{A} \otimes_A D^{(m)}(A), L \otimes_{D^{(m)}(\bar{A})} D^{(m)}_{f\to}) \arrow[d, dash, "\sim"] \\
 & & \mathbf{Mod}_A(\bar{A}, L \otimes_{D^{(m)}(\bar{A})} D^{(m)}_{f\to}) \arrow[d, dash, "\sim"] \\
 & & \mathrm{Ann}_{L\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}} \Bbbk\partial_y^{<i>})}(y) \arrow[d, equal] \\
 & \ell \otimes 1 & \{v \in L \otimes_{\Bbbk} (\coprod_{i\in\mathbb{N}} \Bbbk\partial_y^{<i>}) \mid vy = 0\},
\end{tikzcd}$$

where the structure of left $D^{(m)}(\bar{A})$-module on $\mathbf{Mod}D^{(m)}(A)(D^{(m)}_{f\to}, L \otimes_{D^{(m)}(\bar{A})} D^{(m)}_{f\to})$ is given by

$$\delta \cdot (\ell \otimes ?) = \ell \otimes ((^t\delta)?) \quad \text{with } {}^t\delta \text{ the adjoint of } \delta,$$

$\otimes_{D^{(m)}(\bar{A})}$ is taken with respect to the structure of right $D^{(m)}(\bar{A})$-module on L such that

$\ell\cdot\delta = ({}^t\delta)\ell$. Now

$$
\begin{aligned}
(\ell\otimes\partial_y^{<i>})y &= \ell\otimes\sum_{j\le i}\left\{\begin{matrix} i\\ j\end{matrix}\right\}\partial_y^{<j>}(y)\partial_y^{<i-j>}\\
&= \ell\otimes\sum_{j\le i}\left\{\begin{matrix} i\\ j\end{matrix}\right\}q!\binom{1}{j}y^{1-j}(y)\partial_y^{<i-j>} \quad \text{with } j=p^mq+r, r\in[0,p^m[\\
&= \ell\otimes(\left\{\begin{matrix} i\\ 0\end{matrix}\right\}y\partial_y^{<i>}+\left\{\begin{matrix} i\\ 1\end{matrix}\right\}\partial_y^{<i-1>})\\
&= \left\{\begin{matrix} i\\ 1\end{matrix}\right\}\ell\otimes\partial_y^{<i-1>} \quad \text{as } y=0 \text{ in } \bar{A}\\
&= \begin{cases}\ell\otimes\partial_y^{<i-1>} & \text{if } 1\le i\le p^m-1\\ 0 & \text{if, eg., } i=p^{m+1}.\end{cases}
\end{aligned}
$$

Thus $\ell\otimes\partial_y^{<p^{m+1}>}\in \mathrm{Ann}_{L\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_y^{<i>})}(y)$.

On the other hand, as $\bar{D}^{(m)}(A)=\coprod_{i,j=0}^{p^m-1}A\partial_x^{<i>}\partial_y^{<j>}$, the adjunction for $\bar{D}^{(m)}(\bar{A})$-module reads

$$L\to \mathrm{Ann}_{L\otimes_{\Bbbk}(\coprod_{i=0}^{p^m-1}\Bbbk\partial_y^{<i>})}(y)\simeq L.$$

3° Verma modules

(3.1) Back to the set up of §1, let $\mathcal{B}_w=B^+wB/B$ with B^+ the Borel subgroup opposite to B, and $k_w:\mathcal{B}_w\hookrightarrow\mathcal{B}$. We will abbreviate $\mathcal{D}^{(m)}_{\mathcal{B}}$ as $\mathcal{D}^{(m)}$. $\forall m\in\bar{\mathbb{N}}$, $\mathcal{D}^{(m)}_{k_w\leftarrow}$ is locally free as right $\mathcal{D}^{(m)}_{\mathcal{B}_w}$-module. Then, as k_w is affine, $\int^0_{k_w,(m)}=k_{w*}(\mathcal{D}^{(m)}_{k_w\leftarrow}\otimes_{\mathcal{D}^{(m)}_{\mathcal{B}_w}}?)$ is exact on $\mathrm{qc}(\mathcal{D}^{(m)}_{\mathcal{B}_w})$, so that we may write $\int_{k_w,(m)}$ for $\int^0_{k_w,(m)}$.

If $\overline{\mathcal{B}_w}$ is the closure of $\mathcal{B}_w$ in $\mathcal{B}$, $\partial\mathcal{B}_w=\overline{\mathcal{B}_w}\setminus\mathcal{B}_w$, and if $\ell(w)$ is the length of w, one has [K98, 4.1] as in characteristic 0

(1) $$\mathbb{R}\Gamma_{\overline{\mathcal{B}_w}/\partial\mathcal{B}_w}\simeq\int_{k_w,(m)}\circ\mathbb{L}(k_w^*)[-\ell(w)]:\mathrm{D}^b(\mathrm{qc}(\mathcal{D}^{(\infty)}))\to\mathrm{D}^b(\mathrm{qc}(\mathcal{D}^{(\infty)}));$$

$\forall i\in\mathbb{N}$, $\exists$ isomorphism of B^+-equivariant $\mathcal{D}^{(\infty)}$-modules

(2) $$\mathcal{H}^i_{\overline{\mathcal{B}_w}/\partial\mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})\simeq\begin{cases}\int_{k_w}\mathcal{O}_{\mathcal{B}_w} & \text{if } i=\ell(w)\\ 0 & \text{otherwise;}\end{cases}$$

and $\forall j\in\mathbb{N}$, $\exists$ isomorphism of $\mathrm{Dist}(G)-B^+$-modules

(3) $$\mathrm{H}^i(\mathcal{B},\mathcal{H}^j_{\overline{\mathcal{B}_w}/\partial\mathcal{B}_w}(\mathcal{O}_{\mathcal{B}}))\simeq\begin{cases}\mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B},\mathcal{O}_{\mathcal{B}}) & \text{if } i=0 \text{ and } j=\ell(w)\\ 0 & \text{otherwise.}\end{cases}$$

$\forall \lambda \in \Lambda \simeq \mathbf{GrpSch}(B, GL_1)$, let $\Bbbk_\lambda$ be the 1-dimensional B-module defined by λ and put $\Delta_\infty(\lambda) = \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} \Bbbk_\lambda$. If M is a T-module, we will denote by $\mathrm{ch}\, M = \sum_{\lambda \in \Lambda} \dim(M_\lambda) e(\lambda)$ the formal character of M in the group ring $\mathbb{Z}[\Lambda] = \coprod_{\lambda \in \Lambda} \mathbb{Z} e(\lambda)$ of Λ.

Proposition: *Let $\lambda \in \Lambda$ and $\mathcal{L}(\lambda)$ the invertible $\mathcal{O}_{\mathcal{B}}$-module induced by λ.*

(i) **[K90, 3.1]:** *There is an isomorphism of* $\mathrm{Dist}(G) - T$*-modules*

$$\mathrm{H}^0_{\mathcal{B}_1}(\mathcal{B}, \mathcal{L}(\lambda)) \simeq \Delta_\infty(-\lambda)^\star,$$

where the RHS is the weight-space-wise dual of $\Delta_\infty(-\lambda)$.

(ii) **[K90, 3.2]:** $\mathrm{ch}\, \mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{L}(\lambda)) = \mathrm{ch}\, \Delta_\infty(-w \bullet \lambda)^\star = e(w \bullet \lambda) \prod_{\alpha \in R^+} \dfrac{1}{1 - e(-\alpha)}$.

(iii) **[K90, 3.2]:** *If s is a simple reflexion in W and if $\nu \in \Lambda$, there is an isomorphism of* $\mathrm{Dist}(G)$*-modules* $\mathrm{H}^1_{\mathcal{B}_s}(\mathcal{B}, \mathcal{L}(\lambda)) \simeq \mathrm{H}^0_{\mathcal{B}_1}(\mathcal{B}, \mathcal{L}(\nu))$ *iff $\lambda = s \bullet \lambda = \nu$.*

(iv) **Bøgvad [Bø02]:** *$\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is coherent over $\mathcal{D}^{(\infty)}$.*

(v) *Assume $p > 2(h-1)$. $\forall m \in \mathbb{N}$, $\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is not coherent over $\mathcal{D}^{(m)}$ under $\rho_m : \mathcal{D}^{(m)} \to \mathcal{D}^{(\infty)}$. In particular, $\int_{k_1,(m)} \mathcal{O}_{\mathcal{B}_1} \simeq k_{1*}\mathcal{O}_{\mathcal{B}_1} \simeq \mathcal{H}^0_{\mathcal{B}/\partial \mathcal{B}_1}(\mathcal{O}_{\mathcal{B}})$ is not coherent over $\mathcal{D}^{(m)}$.*

Proof: (v) We have only to show that $\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is not of finite type over $\bar{\mathcal{D}}^{(m)}$. For that, as $\bar{\mathcal{D}}^{(m)}$ is a $\bar{\mathcal{D}}^{(0)}$-module locally of finite type, it is enough to verify that $\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is not coherent over $\bar{\mathcal{D}}^{(0)}$. Just suppose $\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is coherent over $\bar{\mathcal{D}}^{(0)}$. Then by the BMR derived equivalence

$$\begin{aligned} \mathrm{D}^b(\mathbf{U}^0\mathbf{mod}_0) \ni\ & \mathbb{R}\Gamma(\mathcal{B}, \mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})) \\ & \simeq \mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) \quad \text{as } \mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}}) \text{ is } \Gamma(\mathcal{B}, ?)\text{-acyclic by (3).} \end{aligned}$$

It then follows from [BMR, 3.1.6] that$\mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) \in \mathbf{U}^0\mathbf{mod}_0$. Moreover, as $\mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}})$ is a $\bar{\mathcal{D}}^{(0)}$-module, $\mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) \simeq \Gamma(\mathcal{B}, \mathcal{H}^{\ell(w)}_{\overline{\mathcal{B}_w}/\partial \mathcal{B}_w}(\mathcal{O}_{\mathcal{B}}))$ is a $\mathbf{U}_0$-module: under the morphism (1.3.1) one has

$$\begin{tikzcd}
\Gamma(\mathbb{V}(\mathcal{T}_{\mathcal{B}}), \mathcal{O}_{\mathbb{V}(\mathcal{T}_{\mathcal{B}})}) \arrow[d, "\sim"'] & & \Bbbk[\mathcal{N}] \arrow[ll, "p_2^\sharp"'] \\
\Gamma(\mathcal{B}, \mathrm{S}(\mathcal{T}_{\mathcal{B}})) \arrow[d, "\sim"'] & \circlearrowleft & \Bbbk[\mathfrak{g}] \arrow[u, "\mathrm{res}"'] \arrow[d, "\sim"] \\
\mathrm{S}(\mathrm{Der}_{\Bbbk}(\mathcal{O}_{\mathcal{B}})) & & \mathrm{S}(\mathfrak{g}). \arrow[ll, "\mathrm{S}(\mathfrak{g}^{\mathrm{op}}-\mathrm{action\,on}\,\mathcal{O}_{\mathcal{B}})"]
\end{tikzcd}$$

Then $\mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{O}_{\mathcal{B}})$ would be a $\mathbf{U}_0$-module of finite type while $\mathrm{H}^{\ell(w)}_{\mathcal{B}_w}(\mathcal{B}, \mathcal{O}_{\mathcal{B}})$ is infinite dimensional by (ii), absurd.

(3.2) Let $m \in \mathbb{N}$. $\forall w \in W$, let $\mathcal{I}_w$ be the ideal sheaf of $\mathcal{O}_{\mathcal{B}_w}$ defining w and let $\mathcal{O}_{\mathrm{FN}}^{(m)}(w) = \mathcal{O}_{\mathcal{B}_w}/(\mathcal{I}_w^{[m]})$ be the direct image of the structure sheaf of the m-th Frobenius neighbourhood of w in $\mathcal{B}_w$. Put

$$\mathcal{Z}_{w,(m)} = \bar{\mathcal{D}}^{(m)} \otimes_{\mathcal{D}^{(m)}} \int_{k_w,(m)} \mathcal{O}_{\mathrm{FN}}^{(m+1)}(w),$$

$G_m = \ker(\mathrm{Fr}^m : G \to G^{(m)})$ (resp. $B_m = \ker(\mathrm{Fr}^r : B \to B^{(r)})$) the m-th Frobenius kernel of G (resp. B), ${}^wB_m = wB_mw^{-1}$, and

$$\Delta_m(w) = \mathrm{Dist}(G_m) \otimes_{\mathrm{Dist}({}^wB_m)} \Bbbk_{w\bullet 0-(p^m-1)(\rho+w\rho)}.$$

Thus the formal character of $\Delta_m(w)$ is

$$\mathrm{ch}\,\Delta_m(w) = e(w \bullet 0) \prod_{\alpha \in R^+} \frac{1-e(-p^m\alpha)}{1-e(-\alpha)}.$$

Theorem: *Let $m \in \mathbb{N}$.*

(i) $\mathcal{Z}_{w,(m)}$ *is $\Gamma(\mathcal{B}, ?)$-acyclic.*

(ii) $\exists$ *isomorphism of $G_{m+1}T$-modules*

$$\mathrm{R}\Gamma(\mathcal{B}, \mathcal{Z}_{w,(m)}) \simeq \Delta_{m+1}(w).$$

(iii) $\mathcal{Z}_{w,(m)}$ *is irreducible over $\bar{\mathcal{D}}^{(m)}$ with support $\{wB\}$.*

(iv) *Recall from (2.2.ii) that $\mathrm{Z}(\mathcal{D}^{(m)})$ is locally a polynomial algebra over $\mathcal{O}_{\mathcal{B}^{(m+1)}}$ in $\partial_i^{<p^{m+1}>}$, $i \in [1, N]$, $N = |R^+|$. Accordingly, there is a natural morphism of schemes $f : \mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)})) \to \mathcal{B}^{(m+1)}$. Let $\bar{f} : (\mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)})), \mathcal{O}_{\mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)}))}) \to (\mathcal{B}^{(m+1)}, \mathrm{Z}(\mathcal{D}^{(m)}))$ be the induced morphism of ringed spaces. Then $\mathcal{Z}_{w,m}$ is a unique simple $\mathcal{D}^{(m)}$-module of support $\{wB\}$ and supported by $\mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)})/\mathcal{K}_m)$ in $\mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)}))$ through $\bar{f}$, i.e., $\mathrm{supp}(\bar{f}^*(\mathcal{Z}_{w,m})) \subseteq \mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)})/\mathcal{K}_m)$.*

(v) *If $p > 2(h-1)$, under the BMR derived equivalence $\exists$ isomorphism in $\mathrm{D}^b(\mathrm{Coh}(\mathcal{D}^{(0)}))$*

$$\mathcal{Z}_{w,(0)} \simeq \mathcal{D}^{(0)} \otimes^{\mathbb{L}}_{\mathbf{U}^0} \Delta_1(w).$$

Proof: One can show (i)-(iii) and (v) just as in [K?]: by (2.6)

$$\bar{\mathcal{D}}^{(m)} \otimes_{\mathcal{D}^{(m)}} \int_{k_w} \mathcal{O}_{\mathrm{FN}}^{(m+1)}(w) \simeq \bar{\int}_{k_w} \mathcal{O}_{\mathrm{FN}}^{(m+1)}(w).$$

(iv) Let $\mathcal{L}$ be a simple $\mathcal{D}^{(m)}$-module of support $\{wB\}$ such that $\mathrm{supp}(\bar{f}^*(\mathcal{Z}_{w,m})) \subseteq \mathrm{Spec}(\mathrm{Z}(\mathcal{D}^{(m)})/\mathcal{K}_m)$. Consider the adjunction $\mathcal{L} \to j_{w*}j_w^{-1}(\mathcal{L}) \simeq j_{w*}(\mathcal{L}|_{\Omega_w})$. On Ω_w it is invertible: $\mathcal{L}|_{\Omega_w} \simeq \{j_{w*}(\mathcal{L}|_{\Omega_w})\}|_{\Omega_w}$, while on Ω_y, $y \in W \setminus \{w\}$,

$$\Gamma(\Omega_y, \mathcal{L}) \leq \prod_{z \in \Omega_y} \mathcal{L}_z = 0 \quad \text{as } wB \notin \Omega_y;$$

likewise $\Gamma(\Omega_y, j_{w*}(\mathcal{L}|_{\Omega_w})) = \Gamma(\Omega_y \cap \Omega_w, \mathcal{L}) \leq \prod_{z \in \Omega_y} \mathcal{L}_z = 0$. It follows that the adjunction is an isomorphism of $\mathcal{D}^{(m)}$-modules $\mathcal{L} \simeq j_{w*}(\mathcal{L}|_{\Omega_w})$. It thus sffices to show

$$\mathcal{L}|_{\Omega_w} \simeq \bar{\int}_{i_w} \mathcal{O}_{\mathrm{FN}}^{(m+1)}(w).$$

By the irreducibility of $\mathcal{L}$ one must have $\mathcal{L}|_{\Omega_w}$ irreducible over $\mathcal{D}^{(m)}_{\Omega_w}$. Put for simplicity $L = \Gamma(\Omega_w, \mathcal{L})$, $D = \Gamma(\Omega_w, \mathcal{D}^{(m)})$. If $A = \Gamma(\Omega_w, \mathcal{O}_{\mathcal{B}})$ and $N = |R^+|$, by (2.2.ii)

$$\mathrm{Z}(D) = A^{[m+1]}[\partial_i^{<p^{m+1}>} \mid i \in [1, N]].$$

Write $L \simeq D/I$ for some maximal ideal I of D. As D is free over $\mathrm{Z}(D)$ of finite rank by (2.3.2), L is of finite type over $\mathrm{Z}(D)$. Then by [BC, II.4.4 Prop.17]

$$\mathrm{supp}_{\mathrm{Spec}(\mathrm{Z}(D))}(L) = \mathrm{V}(\mathrm{Ann}_{\mathrm{Z}(D)}(L)).$$

Consequently, $\forall i \in [1, N]$, $\exists n_i \in \mathbb{N}$: $(\partial_i^{<p^{m+1}>})^{n_i} L = 0$. Then, in fact, $\partial_i^{<p^{m+1}>} L = 0$ already. For put $\delta = \partial_i^{<p^{m+1}>}$. It is enough to show $\delta D \subseteq I$. Otherwise by the maximality of I

$$D = I + D\delta \quad \text{as } D\delta = \delta D, \ \delta \text{ being central in } D.$$

Thus $\exists \delta_1 \in D$, $\delta_2 \in I$ such that $1 = \delta_2 + \delta_1 \delta$. Then $\delta^{n_i - 1} = \delta^{n_i-1}\delta_2 + \delta_1 \delta^{n_i} \in I$ as $\delta^{n_i} \in I$. It would then follow that $\delta^{n_i-2} = \delta^{n_i-2}\delta_2 + \delta_1 \delta^{n_i-1} \in I$. Repeat to get $1 \in I$, absurd. It follows that L admits a structure of $\bar{D}$-module with $\bar{D} = \Gamma(\Omega_w, \bar{\mathcal{D}}^{(m)})$.

On the other hand, by Cartier-Chase-Smith [H87] $\bar{D}$ is Morita equivalent to $A^{(m+1)}$. Identify Ω_w with $\mathbb{A}^N_{\Bbbk}$ with $wB \mapsto 0$, and write $A = \Bbbk[t] = \Bbbk[t_1, \dots, t_N]$.

By the Nullstellensatz any irreducible $\Bbbk[t]$-module is of the form $\Bbbk[t]/(t_1 - a_1, \dots, t_N - a_N)$, $a_i \in \Bbbk$, nonisomorphic to each other. The corresponding $\bar{D}$-module is $\Bbbk[t] \otimes_{\Bbbk[t]^{(m+1)}} (\Bbbk[t]/(t-a))^{(m+1)}$. But

$$\begin{aligned}
\mathrm{supp}_{\mathbb{A}^N_{\Bbbk}}(\Bbbk[t] \otimes_{\Bbbk[t]^{(m+1)}} (\Bbbk[t]/(t-a))^{(m+1)}) &= \mathrm{V}(\mathrm{Ann}_{\Bbbk[t]}(\Bbbk[t] \otimes_{\Bbbk[t]^{(m+1)}} (\Bbbk[t]/(t-a))^{(m+1)})) \\
&\qquad \text{by [BC, loc. cit.]} \\
&\subseteq \mathrm{V}((t-a)^{p^{m+1}}) \quad \text{as each } t_i^{p^{m+1}} - a_i^{p^{m+1}} = (t_i - a_i)^{p^{m+1}} \text{ annihilates} \\
&\qquad \Bbbk[t] \otimes_{\Bbbk[t]^{(m+1)}} (\Bbbk[t]/(t-a))^{(m+1)} \\
&= \mathrm{V}((t-a)) = \{(t-a)\}.
\end{aligned}$$

Consequently, we must have $L \simeq \Bbbk[t] \otimes_{\Bbbk[t]^{(m+1)}} (\Bbbk[t]/(t))^{(m+1)}$, and hence by the unicity of such

$$\mathcal{L}|_{\Omega_w} \simeq \bar{\int}_{i_w} \mathcal{O}_{\mathrm{FN}}^{(m+1)}(w),$$

as desired.

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