On Kashiwara's equivalence in positive characteristic *

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After the success of Bezrukavnikov, Mirkovic and Rumynin [BMR], we would like to examine some basics of the sheaf $\mathcal{D}_X$ of rings of crystalline differential operators on a smooth variety X over an algebraically closed field $\Bbbk$.

Let $\mathcal{D}iff_X = \mathcal{D}iff_{X/\Bbbk}(\mathcal{O}_X, \mathcal{O}_X)$ be the sheaf of rings of differential operators on $X/\Bbbk$ as defined in [EGAIV]. The sheaf $\mathcal{D}_X$ was introduced by Beilinson and Bernstein [BB], having the presentation

$$\mathcal{D}_X = \mathrm{T}_{\Bbbk}(\mathcal{D}iff_X^1)/ \\ (\lambda - \lambda 1_{\mathcal{O}_X}, a \otimes \delta - a\delta, \delta \otimes \delta' - \delta' \otimes \delta - [\delta, \delta'] \mid \lambda \in \Bbbk, a \in \mathcal{O}_X; \delta, \delta' \in \mathcal{D}iff_X^1),$$

where $\mathcal{D}iff_X^1$ is the sheaf of differential operators of order ≤ 1 in $\mathcal{D}iff_X$ and $\mathrm{T}_{\Bbbk}(\mathcal{D}iff_X^1)$ is the tensor algebra over $\Bbbk$ of $\mathcal{D}iff_X^1$. Thus $\mathcal{D}_X$ coincides with Berthelot's sheaf of PD-differential operators on $X/\Bbbk$ [B] or $\mathcal{D}_X^{(0)}$ in [B96], and is in characteristic 0 just $\mathcal{D}iff_X$.

Let $f : X \to Y$ be a morphism of smooth quasi-projective $\Bbbk$-varieties. One can define the direct image functor $\int_f : \mathrm{D}^b(\mathrm{qc}(\mathcal{D}_X)) \to \mathrm{D}^b(\mathrm{qc}(\mathcal{D}_Y))$ of f from the bounded derived category of quasi-coherent $\mathcal{D}_X$-modules to that of quasi-coherent $\mathcal{D}_Y$-modules just as in characteristic 0. If f is a closed immersion, $\int_f$ is exact, and in characteristic 0 induces an equivalence of category $\mathrm{qc}(\mathcal{D}_X)$ with the full subcategory $\mathrm{qc}_X(\mathcal{D}_Y)$ of $\mathrm{qc}(\mathcal{D}_Y)$ consisting of those with support contained in X [K]. In positive characteristic, however, $\int_f$ no longer induces an equivalence, as opposed to the direct image functor defined for $\mathcal{D}iff_X$ and $\mathcal{D}iff_Y$ by Haastert [H]. Nevertheless, the image of $\mathcal{D}_X$ in $\mathcal{D}iff_X$ under the natural morphism is the first term of the p-filtration on $\mathcal{D}iff_X$ and is a central reduction, denoted $\bar{\mathcal{D}}_X$, of $\mathcal{D}_X$. According to [H] the direct image functor $\bar{\int}_f$ for $\bar{\mathcal{D}}_X$ and $\bar{\mathcal{D}}_Y$ gives an equivalence of category $\mathrm{qc}(\bar{\mathcal{D}}_X)$ to the full subcategory of $\mathrm{qc}(\bar{\mathcal{D}}_Y)$ consisting of those annihilated by the p-th power of the ideal sheaf of $\mathcal{O}_Y$ defining X.

We will show that for Y a flag variety G/B, G a simple algebraic group over $\Bbbk$, B a Borel subgroup of G, and X a Chevalley-Bruhat cell in Y, the direct image $\bar{\int}_f$ of

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the structure sheaf of the Frobenius neighbourhood of a point in X is a coherent $\bar{\mathcal{D}}_{G/B}$-module and corresponds under the Bezrukavnikov-Mirkovic-Rumynin derived equivalence, i.e., by taking global sections, to an infinitesimal Verma module, a standard object in the representation theory of G; the determination of the composition factor multiplicities of those infinitesimal Verma modules as G_1T-modules, G_1 the Frobenius kernel of G and T a maximal torus of B, obtains the characters of all simple G-modules for $p \geq h$ the Coxeter number of G. We find, however, that those direct image $\bar{\mathcal{D}}_{G/B}$-modules are simple as $\bar{\mathcal{D}}_{G/B}$-modules.

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1° **Kashiwara's equivalence**

Throughout the rest of the paper $\Bbbk$ will be assumed to have characteristic $p > 0$. By $\otimes_X$ we will mean the tensor product over the structure sheaf $\mathcal{O}_X$ of X. For a category $\mathcal{C}$ and its objects C_1, C_2 we will denote the set of morphisms in $\mathcal{C}$ from C_1 to C_2 by $\mathcal{C}(C_1, C_2)$. Let $f : X \to Y$ be a morphism of smooth quasi-projective $\Bbbk$-varieties.

(1.1) Let us briefly recall from [B00] the construction of the direct image functor $\int_f$. The inverse image functor $f^* : \mathrm{qc}(\mathcal{D}_Y) \to \mathrm{qc}(\mathcal{D}_X)$ defines a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule $\mathcal{D}_{f\to} = f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ with the structure of right $f^{-1}\mathcal{D}_Y$-module provided by the multiplication to the right on $f^{-1}\mathcal{D}_Y$.

If $\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X)$ is the category of quasi-coherent right $\mathcal{D}_X$-modules, the direct image functor $f_+^{\mathrm{rgt}} : \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X) \to \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y)$ for right modules is defined by $f_+^{\mathrm{rgt}} = f_*(? \otimes_{\mathcal{D}_X} \mathcal{D}_{f\to})$, using the structure of right $f^{-1}\mathcal{D}_Y$-module on $\mathcal{D}_{f\to}$. If ω_X is the dualizing sheaf on X, ω_X is equipped with a structure of right $\mathcal{D}_X$-module and define an equivalence of categories $\omega_X \otimes_X ? : \mathrm{qc}(\mathcal{D}_X) \to \mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X)$ with quasi-inverse $? \otimes_X \omega_X^{-1}$. Then the direct image functor $\int_f^0 : \mathrm{qc}(\mathcal{D}_X) \to \mathrm{qc}(\mathcal{D}_Y)$ is defined by

$$\int_f^0 = (? \otimes_Y \omega_Y^{-1}) \circ f_+^{\mathrm{rgt}} \circ (\omega_X \otimes_X ?).$$

Alternatively, $f^*(\mathcal{D}_Y \otimes_Y \omega_Y^{-1})$ is equipped with two isomorphic natural structures of left $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-modules [B00, 3.4.1], and defines a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\mathcal{D}_{f\leftarrow} = \omega_X \otimes_X f^*(\mathcal{D}_Y \otimes_Y \omega_Y^{-1})$. Then

$$\int_f^0 \simeq f_*(\mathcal{D}_{f\leftarrow} \otimes_{\mathcal{D}_X} ?).$$

In the derived category we set $\int_f = \mathrm{R}f_*(\mathcal{D}_{f\leftarrow} \otimes^{\mathrm{L}}_{\mathcal{D}_X} ?) : \mathrm{D}^b(\mathrm{qc}(\mathcal{D}_X)) \to \mathrm{D}^b(\mathrm{qc}(\mathcal{D}_Y))$.

In case f is an open immersion, $\int_f \simeq \mathrm{R}f_*$. If $g : Y \to Z$ is another morphism of smooth quasi-projective $\Bbbk$-varieties, $\int_{g\circ f} \simeq \int_g \circ \int_f$.

(1.2) Assume from now on that f is a closed immersion. Let us describe the local structure of $\mathcal{D}_{f\to}$ and $\mathcal{D}_{f\leftarrow}$. Let $\mathfrak{Y}$ be an affine open of Y with coordinate $(t_1,\dots,t_r,t_{r+1},\dots,t_{r+s})$ such that $f^{-1}\mathfrak{Y}$ is defined by the ideal $(t_1,\dots,t_r)$ and is equipped with coordinate $(\bar t_{r+1},\dots,\bar t_{r+s})$, where $\bar t_i$ is the image of t_i in $\Bbbk[f^{-1}\mathfrak{Y}]$. Put $A=\Bbbk[\mathfrak{Y}]$, $D(A)=\Gamma(\mathfrak{Y},\mathcal{D}_Y)$, $\partial_j=\frac{\partial}{\partial t_j}\in D(A), j\in[1,r+s]$, $\bar A=\Bbbk[f^{-1}\mathfrak{Y}]$, $D(\bar A)=\Gamma(f^{-1}\mathfrak{Y},\mathcal{D}_X)$, $\bar\partial_i=\frac{\partial}{\partial \bar t_i}\in D(\bar A), i\in[r+1,r+s]$, $D_{f\to}=\Gamma(f^{-1}\mathfrak{Y},\mathcal{D}_{f\to})$, $D_{f\leftarrow}=\Gamma(f^{-1}\mathfrak{Y},\mathcal{D}_{f\leftarrow})$. Then the structure of $(D(\bar A),D(A))$-bimodule on $D_{f\to}\simeq\bar A\otimes_A D(A)$ is given by

$$(1)\qquad (\bar a\bar\partial_i)\cdot(1\otimes\delta)\cdot\delta'=1\otimes a\partial_i\delta\delta'\quad \forall a\in A\,\forall\delta,\delta'\in D(A)\,\forall i\in[r+1,r+s],$$

where $\bar a$ is the image of a in $\bar A$. It follows that $D_{f\to}$ is free over $D(\bar A)$ of basis $\partial^n, n\in\mathbb{N}^r$:

$$(2)\qquad D_{f\to}\simeq\bar A\otimes_A(\coprod_{k\in\mathbb{N}^{r+s}}A\partial^k)\simeq(\coprod_{\ell\in\mathbb{N}^s}\bar A\bar\partial^\ell)\otimes_\Bbbk(\coprod_{n\in\mathbb{N}^r}\Bbbk\partial^n)\simeq D(\bar A)\otimes_\Bbbk(\coprod_{n\in\mathbb{N}^r}\Bbbk\partial^n),$$

where $\partial^k=\partial_1^{k_1}\dots\partial_{r+s}^{k_{r+s}}$, etc., and the structure of $D(\bar A)$-module on the RHS is given by the multiplication to the left on $D(\bar A)$. On the other hand, the structure of $(D(A),D(\bar A))$-bimodule on $D_{f\leftarrow}\simeq\bar A\otimes_A D(A)$ is given by

$$(3)\qquad \delta'\cdot(1\otimes\delta)\cdot(\bar a\bar\partial_i)=1\otimes{}^t(a\partial_i)\delta({}^t\delta')\quad\forall a\in A\,\forall\delta,\delta'\in D(A)\,\forall i\in[r+1,r+s],$$

where ${}^t(a\partial_i)=-\partial_i a$ is the adjoint of $a\partial_i$. Thus $D_{f\leftarrow}$ is also free over $D(\bar A)$ of basis $\partial^n, n\in\mathbb{N}^r$:

$$(4)\qquad D_{f\leftarrow}\simeq(\coprod_{n\in\mathbb{N}^r}\Bbbk\partial^n)\otimes_\Bbbk D(\bar A),$$

where the structure of $D(\bar A)$-module on the RHS is given by the multiplication to the right on $D(\bar A)$. In particular, $\int_f^0$ is exact, and hence we may abbreviate $\int_f^0$ as $\int_f$.

Define another functor $f^+_{\mathrm{rgt}}:\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y)\to\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X)$ via $\mathcal{M}\mapsto\mathcal{M}od(f^{-1}\mathcal{D}_Y)(\mathcal{D}_{f\to},f^{-1}\mathcal{M})$ the sheaf of morphisms of right $f^{-1}\mathcal{D}_Y$-modules from $\mathcal{D}_{f\to}$ to $f^{-1}\mathcal{M}$. If $\mathrm{qc}^{\mathrm{rgt}}_X(\mathcal{D}_Y)$ is the full subcategory of $\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_Y)$ consisting of those with support $\subseteq X$, then as in characteristic 0 or as for the ordinary differential operators $\mathcal{D}iff$ in place of $\mathcal{D}$, f_+^{rgt} is left adjoint to $f^+_{\mathrm{rgt}}\mid_{\mathrm{qc}_X^{\mathrm{rgt}}(\mathcal{D}_Y)}$ [H, 8.4]. Accordingly, if $f^+=(?\otimes_X\omega_X^{-1})\circ f^+_{\mathrm{rgt}}\circ(\omega_Y\otimes_Y ?)\simeq(f^{-1}\mathcal{D}_Y)\mathcal{M}od(\mathcal{D}_{f\leftarrow},f^{-1}?):\mathrm{qc}(\mathcal{D}_Y)\to\mathrm{qc}(\mathcal{D}_X)$, then $\int_f$ is left adjoint to $f^+\mid_{\mathrm{qc}_X(\mathcal{D}_Y)}$.

(1.3) Keep the notations of (1.2), but with $r=1$ for simplicity. Let $\mathcal{M}\in\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X)$, $\mathcal{L}\in\mathrm{qc}_X^{\mathrm{rgt}}(\mathcal{D}_Y)$, and put $M=\Gamma(f^{-1}\mathfrak{Y},\mathcal{M})$, $L=\Gamma(\mathfrak{Y},\mathcal{L})$. Then the adjunctions $\mathrm{id}_{\mathrm{qc}^{\mathrm{rgt}}(\mathcal{D}_X)}\to f^+_{\mathrm{rgt}}\circ f_+^{\mathrm{rgt}}$ and $f_+^{\mathrm{rgt}}\circ f^+_{\mathrm{rgt}}\mid_{\mathrm{qc}_X^{\mathrm{rgt}}(\mathcal{D}_Y)}\to\mathrm{id}_{\mathrm{qc}_X^{\mathrm{rgt}}(\mathcal{D}_Y)}$ read for $\mathcal{M}$ on $f^{-1}\mathfrak{Y}$ and $\mathcal{L}$ on $\mathfrak{Y}$, respectively, as the following commutative diagrams

$$(1)\qquad \begin{tikzcd}
m \arrow[rr, mapsto] & & m\otimes ? \\
M \arrow[rr] \arrow[ddrr, dashed, start anchor=south east, end anchor=west] & & \mathbf{Mod}D(A)(D_{f\to},M\otimes_{D(\bar A)}D_{f\to}) \arrow[d, "\wr"] \\
m \arrow[ddr, mapsto] & & \mathbf{Mod}D(A)(\bar A\otimes_A D(A),M\otimes_{D(\bar A)}(D(\bar A)\otimes_\Bbbk\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i)) \arrow[d, "\wr"] \\
& & \mathbf{Mod}A(\bar A,M\otimes_\Bbbk\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i) \arrow[d, "\wr"] \\
& m\otimes 1 & \mathrm{Ann}_{M\otimes_\Bbbk(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i)}(t_1),
\end{tikzcd}$$

where $\mathrm{Ann}_{M\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i)}(t_1)$ is the annihilator of t_1 in $M\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i)$, and

$$
\begin{tikzcd}
\phi\otimes\eta \arrow[r, mapsto] & \phi(\eta) & \\
\mathbf{Mod}D(A)(D_{f\to},L)\otimes_{D(\bar A)}D_{f\to} \arrow[r] & L & v\partial_1^i. \\
\mathrm{Ann}_L(t_1)\otimes_{\Bbbk}(\coprod_i\Bbbk\partial_1^i) \arrow[u, "\sim"] \arrow[ur, dashed] & v\otimes\partial_1^i \arrow[ur, mapsto] &
\end{tikzcd}
\tag{2}
$$

But $\forall m\in M$ one has in $M\otimes_{\Bbbk}(\coprod_i\Bbbk\partial_1^i)\simeq M\otimes_{D(\bar A)}(\coprod_i D(\bar A)\partial_1^i)$

$$
\begin{aligned}
(m\otimes\partial_1^i)\cdot t_1 &= m\otimes\partial_1^i t_1\\
&= m\otimes\sum_{j=0}^{i}\binom{i}{j}\partial_1^j(t_1)\partial_1^{i-j}\quad\text{by [B96, 2.2.4]}\\
&= m\otimes(t_1\partial_1^i+i\partial_1^{i-1})\\
&= m\otimes i\partial_1^{i-1}\quad\text{as } t_1=0 \text{ in } \bar A,
\end{aligned}
$$

hence $\mathrm{Ann}_{M\otimes_{\Bbbk}(\coprod_i\Bbbk\partial_1^i)}(t_1)=M\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^{pi})$. It follows that $\mathcal{M}\to(f^+_{\mathrm{rgt}}\circ f_+^{\mathrm{rgt}})(\mathcal{M})$ cannot be epic unless $\mathcal{M}=0$.

Take $\mathcal{L}=f_+^{\mathrm{rgt}}(\mathcal{M})$. Then (2) reads as

$$
\begin{tikzcd}
\mathbf{Mod}D(A)(D_{f\to},L)\otimes_{D(\bar A)}D_{f\to} \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & L \\
(M\otimes_{\Bbbk}\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^{pi})\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i) \arrow[u, "\sim"] \arrow[r, dashed] & M\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial_1^i) \arrow[u, "\sim"'] \\
m\otimes\partial_1^{pi}\otimes\partial_1^j \arrow[r, mapsto] & m\otimes\partial_1^{pi+j},
\end{tikzcd}
\tag{3}
$$

which fails to be monic unless $M=0$. Consider also the case $A=\Bbbk[t]$, $\bar A=\Bbbk[t]/(t)\simeq\Bbbk$, and $L=\Bbbk[t]/(t^{2p})$. As t^{2p} is central in $D(\Bbbk[t])$, L admits a structure of right $D(\Bbbk[t])$-module :

$$a\cdot\delta=({}^t\delta)(a)\quad\forall a\in L,\delta\in D(A).$$

Then (2) reads as

$$(t^{2p-1}L)\otimes_{\Bbbk}(\coprod_{i\in\mathbb{N}}\Bbbk\partial^i)\to L\quad\text{via}\quad a\otimes\partial^j\mapsto(-1)^j\partial^j(a),\tag{4}$$

which fails to be surjective.

If f_+^{rgt} were to be an equivalence to a subcategory $\mathcal{C}$ of $\mathrm{qc}_X(\mathcal{D}_Y)$, its quasi-inverse should be its right adjoint $f^+_{\mathrm{rgt}}\mid_{\mathcal{C}}$. We thus find a failure of Kashiwara's equivalence :

Proposition. $\int_f:\mathrm{qc}(\mathcal{D}_X)\to\mathrm{qc}(\mathcal{D}_Y)$ *does not induce an equivalence of categories unless* f *is an isomorphism.*

(1.4) Recall from Haastert [H], however, that the direct image functor from $\mathrm{qc}(\mathcal{D}iff_X)$ to $\mathrm{qc}(\mathcal{D}iff_Y)$ induces an equivalence and likewise for the r-th terms $\mathcal{M}od_{\mathcal{O}_X^{(r)}}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathcal{M}od_{\mathcal{O}_Y^{(r)}}(\mathcal{O}_Y, \mathcal{O}_Y)$, $r \in \mathbb{N}$, of the p-filtration on $\mathcal{D}iff_X$ and on $\mathcal{D}iff_Y$, respectively, where $\mathcal{O}_X^{(r)} = \{a^{p^r} \mid a \in \mathcal{O}_X\}$ and likewise $\mathcal{O}_Y^{(r)}$. After [H] we will denote $\mathcal{M}od_{\mathcal{O}_X^{(r)}}(\mathcal{O}_X, \mathcal{O}_X)$ by $\mathcal{D}_{X,r}$. The direct image functor

$$\int_{f,r}^0 = f_*(\mathcal{D}_{f\leftarrow,r} \otimes_{\mathcal{D}_{X,r}} ?) : \mathrm{qc}(\mathcal{D}_{X,r}) \to \mathrm{qc}(\mathcal{D}_{Y,r}) \quad \text{with} \quad \mathcal{D}_{f\leftarrow,r} = \omega_X \otimes_X f^*(\mathcal{D}_{Y,r} \otimes_Y \omega_Y^{-1})$$

is exact, and an equivalence to the full subcategory of $\mathrm{qc}(\mathcal{D}_{Y,r})$ consisting of those annihilated by $(a^{p^r} \mid a \in \mathcal{I}_X)$, $\mathcal{I}_X$ the ideal of $\mathcal{O}_Y$ defining X.

Back to the general $f : X \to Y$ and $g : Y \to Z$ two morphisms of smooth quasi-projective $\Bbbk$-varieties, recall also that

$$\int_{g\circ f,r} = \int_{g,r} \circ \int_{f,r} \tag{1}$$

with $\int_{f,r} = \mathrm{R}f_*(\mathcal{D}_{f\leftarrow,r} \otimes_{\mathcal{D}_{X,r}} ?)$, etc.; $\mathcal{D}_{f\leftarrow,r}$ is a projective right $\mathcal{D}_{X,r}$-module. In case f is an open immersion $\int_{f,r} \simeq \mathrm{R}f_*$ again. The image of $\mathcal{D}_X$ in $\mathcal{D}iff_X$ under the natural morphism is $\mathcal{D}_{X,1} = \mathcal{M}od_{\mathcal{O}_X^{(1)}}(\mathcal{O}_X, \mathcal{O}_X)$, and is isomorphic to a central reduction of $\mathcal{D}_X$

$$\bar{\mathcal{D}}_X = \mathcal{D}_X \otimes_{\mathfrak{Z}_X} \{\mathrm{S}_X(\mathcal{T}_X)^{(1)}/(\mathcal{T}_X^{(1)})\},$$

where $\mathfrak{Z}_X$ is the center of $\mathcal{D}_X$, which is isomorphic to $\mathrm{S}_X(\mathcal{T}_X)^{(1)}$ with $\mathcal{T}_X$ the tangent sheaf of X [BMR, 1.3.3]:

$$\begin{tikzcd} \mathcal{D}_X \arrow[rr] \arrow[d] \arrow[drr, phantom, "\circlearrowleft"] & & \mathcal{D}iff_X \\ \bar{\mathcal{D}}_X \arrow[rr, dashed, "\sim"'] & & \mathcal{M}od_{\mathcal{O}_X^{(1)}}(\mathcal{O}_X, \mathcal{O}_X). \arrow[u, hook] \end{tikzcd} \tag{2}$$

Thus we will write $\bar{\int}_f^0$ for $\int_{f,1}^0$.

2° Verma modules

In this section we let $\mathcal{B} = G/B$ with G a simple algebraic group over $\Bbbk$ and B a Borel subgroup of G, T a maximal torus of B, U^- the unipotent radical of B, R the root system of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, $W = \mathrm{N}_G(T)/T$ the Weyl group of G, B^+ the Borel subgroup of G opposite to B, U^+ the unipotent radical of B^+, Λ the charactrer group of T, G_1 (resp. B_1^+) the Frobenius kernel of G (resp. B^+), $\mathrm{Dist}(G_1) = \Bbbk[G_1]^*$ the algebra of distributions of G_1, etc., and $\mathfrak{U}$ the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.

For $w \in W$ let $\Omega_w = wU^+B/B$, $\mathcal{B}_w = U^+wB/B$, and let $i_w : \mathcal{B}_w \hookrightarrow \Omega_w$ (resp. $j_w : \Omega_w \hookrightarrow \mathcal{B}$) be the closed (resp. open) immersion. As j_w is affine, $\int_{j_w}^0 = (j_w)_* = \bar{\int}_{j_w}^0$ is exact, so therefore are $\int_{j_w\circ i_w}^0 = \int_{j_w}^0 \circ \int_{i_w}^0$ and $\bar{\int}_{j_w\circ i_w}^0$; we may thus drop superscripts 0 from those. Put $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$.

(2.1) Let us begin with $G = SL_2$, B consisting of lower triangular matrices and T of diagonals. There is an isomorphism of $\Bbbk$-varieties from $\mathcal{B}$ to the projective 1-space $\mathbb{P}^1$ via $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [b, d]$. Let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and put $\Bbbk[x] = \Gamma(\Omega_1, \mathcal{O}_{\mathcal{B}})$ with $x : \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mapsto a$, and $\Bbbk[y] = \Gamma(\Omega_s, \mathcal{O}_{\mathcal{B}})$ with $y : \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix} \mapsto -a$. Then

$$\Gamma(\mathcal{B}, \int_{j_1 \circ i_1} \mathcal{O}_{\mathcal{B}_1}) \simeq \Bbbk[x] \quad \text{while} \quad \Gamma(\mathcal{B}, \int_{j_s \circ i_s} \mathcal{O}_{\mathcal{B}_s}) \simeq \coprod_{i \in \mathbb{N}} \Bbbk \partial_y^i.$$

As $\Gamma(\Omega_s, \int_{j_1 \circ i_1} \mathcal{O}_{\mathcal{B}_1}) \simeq \Bbbk[y, \frac{1}{y}]$ is not of finite type over $\Gamma(\Omega_s, \mathcal{D})$, unlike in characteristic 0

$$\int_{j_1 \circ i_1} \mathcal{O}_{\mathcal{B}_1} \text{ is not coherent over } \mathcal{D}. \tag{1}$$

If ($e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$) is the standard basis of $\mathfrak{g}$, the structure of $\mathfrak{U}$-module on $\Bbbk[x]$ and on $\coprod_i \Bbbk \partial_y^i$ are given, respectively, through

$$\mathfrak{U} \to \Gamma(\Omega_1, \mathcal{D}) \quad \text{such that} \quad e \mapsto -\partial_x, h \mapsto -2x\partial_x, f \mapsto x^2 \partial_x,$$

and

$$\mathfrak{U} \to \Gamma(\Omega_s, \mathcal{D}) \quad \text{such that} \quad e \mapsto y^2 \partial_y, h \mapsto 2y\partial_y, f \mapsto -\partial_y.$$

Then

$$\mathrm{soc}_{\mathfrak{U}}(\Bbbk[x]) = \coprod_{i \in \mathbb{N}} \Bbbk x^{pi} \quad \text{with } \mathfrak{U} \text{ acting trivially on each } \Bbbk x^{pi}, \tag{2}$$

and

$$\Bbbk[x]/\mathrm{soc}_{\mathfrak{U}}(\Bbbk[x]) \simeq \coprod_{i \in \mathbb{N}} \{\mathfrak{U} x^{pi+1}/(\Bbbk x^{pi} \oplus \Bbbk x^{p(i+1)})\} \quad \text{with each } \mathfrak{U} x^{pi+1} = \sum_{j=0}^{p} \Bbbk x^{pi+j}. \tag{3}$$

One has $\mathfrak{U}/(x^p - x^{[p]} \mid x \in \mathfrak{g}) \simeq \mathrm{Dist}(G_1)$, and there are isomorphisms of $G_1 T$-modules

$$\begin{aligned} \mathfrak{U} x^{pi+1}/\Bbbk x^{pi} &\simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} (-2pi - 2), \\ \mathfrak{U} x^{pi+1}/\Bbbk x^{p(i+1)} &\simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1)} (-2p(i+1) + 2), \end{aligned} \tag{4}$$

where we identify Λ with $\mathbb{Z}$ and each $k \in \mathbb{Z}$ denotes the corresponding 1-dimensional $B_1^{\pm} T$-module.

On the other hand, if $\mathfrak{U}(\mathfrak{b}^+)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{b}^+$ of B^+,

$$\coprod_{i \in \mathbb{N}} \Bbbk \partial_y^i \simeq \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b}^+)} (-2), \tag{5}$$

which admits a filtration of $\mathfrak{U}$-modules

$$\coprod_{i \in \mathbb{N}} \Bbbk \partial_y^i > \coprod_{i \geq p-1} \Bbbk \partial_y^i > \coprod_{i \geq p} \Bbbk \partial_y^i > \coprod_{i \geq 2p-1} \Bbbk \partial_y^i \ \dots$$

such that $\forall n \in \mathbb{N}$,

(6)
$$\coprod_{i \geq np} \Bbbk\partial_y^i \simeq \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b}^+)} (-2pn-2),$$
$$\coprod_{i \geq (n+1)p-1} \Bbbk\partial_y^i \simeq \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b}^+)} (-2p(n+1)),$$
$$(\coprod_{i \geq np} \Bbbk\partial_y^i)/(\coprod_{i \geq (n+1)p} \Bbbk\partial_y^i) \simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} (-2pn-2),$$
$$(\coprod_{i \geq (n+1)p-1} \Bbbk\partial_y^i)/(\coprod_{i \geq (n+2)p-1} \Bbbk\partial_y^i) \simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} (-2p(n+1)).$$

(2.2) Back to the general setup, if U_α is the root subgroup of G corresponding to $\alpha \in R$, the group multiplication induces isomorphisms of $\Bbbk$-varieties

(1)
$$U^+ \simeq \prod_{\alpha \in R^+} U_\alpha.$$

Put $\Bbbk[U_\alpha] = \Bbbk[x_\alpha]$, $\partial_\alpha = \dfrac{\partial}{\partial x_\alpha}$, $\partial_\alpha^{(n)} = \dfrac{\partial_\alpha^n}{n!}$, $n \in [0, p[$, and $\bar{\mathcal{D}}(\Omega_1) = \Gamma(\Omega_1, \bar{\mathcal{D}})$. Under the identification (1) we may write

$$\bar{\mathcal{D}}(\Omega_1) = \coprod_{n \in [0,p[^{R^+}} \Bbbk[x_\alpha \mid \alpha \in R^+]\partial^n \quad \text{with} \quad \partial^n = \prod_{\alpha \in w(R^+)} \partial_\alpha^{n_\alpha}.$$

Let $\mu_\alpha^{(i)} \in \mathrm{Dist}(U_\alpha)$, $i \in \mathbb{N}$, such that $\mu_\alpha^{(i)}(x_\alpha^j) = \delta_{ij}$ for each $j \in \mathbb{N}$.

Lemma. *Under the $\Bbbk$-algebra homomorphism $\mathrm{Dist}(U_1^+) \to \bar{\mathcal{D}}(\Omega_1)$ induced by the G-action on $\mathcal{B}$ from the left*
$$\prod_{\alpha \in R^+} \mu_\alpha^{(p-1)} \mapsto \prod_{\alpha \in R^+} \partial_\alpha^{(p-1)}.$$

Proof: Let $\pi_1, \dots, \pi_\ell$ be the simple roots of R^+ and define the height of each $\alpha \in R^+$ to be $\mathrm{ht}(\alpha) = \sum_{i=1}^{\ell} n_i$ if $\alpha = \sum_{i=1}^{\ell} n_i\pi_i$. Enumerate R^+ such that $\mathrm{ht}(\alpha_i) \not> \mathrm{ht}(\alpha_j)$ if $i < j$. Accordingly, write x_i (resp. ∂_i, $\mu_i^{(j)}$) for x_{α_i} (resp. ∂_{α_i} $\mu_\alpha^{(j)}$). By Chevalley's commutator formulae the comorphism $\Bbbk[x_j \mid j \in [1, N]] \to \Bbbk[x_i] \otimes_\Bbbk \Bbbk[x_j \mid j \in [1, N]]$ of the U_{α_i}-action on $\prod_{j=1}^{N} U_{\alpha_j}$ reads

$$x_j \mapsto \begin{cases} 1 \otimes x_j & \text{if } j < i \\ x_i \otimes 1 + 1 \otimes x_i & \text{if } j = i. \end{cases}$$

Thus the homomorphism $\mathrm{Dist}(U_1^+) \to \bar{\mathcal{D}}(\Omega_1)$ sends $\mu_i^{(1)}$ to $-\partial_i + \sum_{j>i} a_{ij}\partial_j$ for some $a_{ij} \in \Bbbk[x_k \mid k \in [1, N]]$. As $\partial_j^p = 0$ for any j in $\bar{\mathcal{D}}(\Omega_1)$ and as $\mu_N^{(p-1)} \mapsto \partial_N^{(p-1)}$,

$$\prod_{i=1}^{N} \mu_i^{(p-1)} \mapsto \prod_{i=1}^{N} \frac{(-\partial_i + \sum_{j>i} a_{ij}\partial_j)^{p-1}}{(p-1)!} = \prod_{i=1}^{N} \partial_i^{(p-1)}.$$

The same argument yields $\prod_{i=1}^N \mu_i^{(p-1)}$ is independent of the order as $\mu_j^{(1)}\mu_j^{(p-1)} = 0\ \forall j$ in $\mathrm{Dist}(U_1^+)$.

(2.3) For each $w \in W$ let $\mathcal{I}_w$ be the ideal sheaf of $\mathcal{O}_{\mathcal{B}_w}$ defining wB in $\mathcal{B}_w$ and let $\mathrm{FN}(w) = \mathrm{FN}_{\mathcal{B}_w}(wB)$ be the infinitesimal Frobenius neighbourhood of wB in $\mathcal{B}_w$ defined by $(a^p \mid a \in \mathcal{I}_w)$. If $k_w : \mathrm{FN}(w) \hookrightarrow \mathcal{B}_w$, $k_{w*}\mathcal{O}_{\mathrm{FN}(w)}$ is equipped with a structure of T-equivariant coherent $\bar{\mathcal{D}}_{\mathcal{B}_w}$-module.

Proposition. *Let* $\Delta_w = \mathrm{R}\Gamma(\mathcal{B}, \bar{\int}_{j_w\circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) \simeq \Gamma(\mathcal{B}, \bar{\int}_{j_w\circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)})$.

(i) *There is an isomorphism of G_1T-modules*

$$\Delta_w \simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}({}^wB_1)} (w \cdot 0 - (p-1)(\rho + w\rho)),$$

where ${}^wB_1$ is the Frobenius kernel of ${}^wB = wBw^{-1}$ and $w \cdot 0 = w\rho - \rho$ with $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

(ii) *If $\bar{\mathcal{D}}(\Omega_w) = \Gamma(\Omega_w, \bar{\mathcal{D}})$, Δ_w is simple as $\bar{\mathcal{D}}(\Omega_w)$-module.*

(iii) *$\bar{\int}_{j_w\circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}$ is coherent over $\bar{\mathcal{D}}$ with* $\mathrm{supp}(\bar{\int}_{j_w\circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) = \{wB\}$.

Proof: Let $U_w^\pm = ({}^wU^+) \cap U^\pm$ with ${}^wU^+ = wU^+w^{-1}$, and $U_{w,1}^+$ the Frobenius kernel of U_w^+. One has a commutative diagram of $\Bbbk$-varieties

(1)
$$\begin{tikzcd}
x \arrow[r, mapsto] & xw \\
{}^wU^+ \arrow[r, "\sim"] & \Omega_w \\
U_w^+ \arrow[u, hook] \arrow[r, "\sim"] & \mathcal{B}_w. \arrow[u, hook] \\
U_{w,1}^+ \arrow[u, hook] \arrow[r, "\sim"'] & \mathrm{FN}(w). \arrow[u, hook]
\end{tikzcd}$$

Let $R_w^\pm = \{w\alpha \gtrless 0 \mid \alpha \in R^+\}$. If U_α is the root subgroup of G corresponding to $\alpha \in R$, the group multiplication induces isomorphisms of $\Bbbk$-varieties

(2)
$${}^wU^+ \simeq \prod_{\alpha\in w(R^+)} U_\alpha, \quad U_w^\pm \simeq \prod_{\alpha\in R_w^\pm} U_\alpha.$$

Put $\Bbbk[U_\alpha] = \Bbbk[x_\alpha]$, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, and $\bar{\mathcal{D}}(\Omega_w) = \Gamma(\Omega_w, \bar{\mathcal{D}})$. Under the identification (2)

$$\bar{\mathcal{D}}(\Omega_w) = \coprod_{n\in[0,p[^{w(R^+)}} \Bbbk[{}^wU^+]\partial^n = \coprod_{j\in[0,p[^{R_w^-}, k\in[0,p[^{R_w^+}} \Bbbk[{}^wU^+]\partial^j\partial^k$$

with $\partial^n = \prod_{\alpha\in w(R^+)} \partial_\alpha^{n_\alpha}$, etc. By (1.2.3) there is an isomorphism of $\bar{\mathcal{D}}(\Omega_w)$-modules

(3)
$$\Delta_w \simeq (\coprod_{n\in[0,p[^{R_w^-}} \Bbbk\partial^n) \otimes_\Bbbk (\Bbbk[x_\alpha \mid \alpha \in R_w^+]/(x_\alpha^p \mid \alpha \in R_w^+))$$

with the structure of $\bar{\mathcal{D}}(\Omega_w)$-module on the RHS given by

$$(4) \qquad (b\partial^j\bar{\partial}^k)\cdot(\partial^n\otimes a) = (-1)^{|j|}\sum_{i\leq j+n}\binom{j+n}{i}\partial^{j+n-i}\otimes\overline{\partial^i(b)}\bar{\partial}^k(a)$$

$\forall a\in \Bbbk[U^+_{w,1}]$ $\forall b\in\Bbbk[{}^wU^+]$ $\forall k\in[0,p[^{R^+_w}$, $\forall j,n\in[0,p[^{R^-_w}$, where $|j|=\sum_{\alpha\in R^-_w} j_\alpha$.

Let us now consider the T-module structure on Δ_w. If $\omega_{U^+_w}$ (resp. $\omega_{\mathcal{B}}$) is the dualizing sheaf on U^+_w (resp. $\mathcal{B}$) and if $\omega(U^+_w)$ (resp. $\omega(\Omega_w)$) is a $\Bbbk[U^+_w]$ (resp. $\Bbbk[\Omega_w]$)-basis of $\Gamma(U^+_w,\omega_{U^+_w})$ (resp. $\Gamma(\Omega_w,\omega_{\mathcal{B}})$), then the T-weights of $\omega(U^+_w)$ and $\omega(\Omega_w)$ relevant to Δ_w are

$$(5) \qquad \mathrm{wt}(\omega(U^+_w)) = -\sum_{\alpha\in R^+_w}\alpha,\quad \mathrm{wt}(\omega(\Omega_w)) = -\sum_{\alpha\in R^+} w\alpha.$$

It follows that in Δ_w

$$(6) \qquad \mathrm{wt}(\partial^n\otimes x^m) = (\sum_{\alpha\in R^-_w} n_\alpha\alpha) - (\sum_{\beta\in R^+_w} m_\beta\beta) + w\cdot 0 \quad \forall n\in[0,p[^{R^-_w}\ \forall m\in[0,p[^{R^+_w}.$$

Therefore the formal character of Δ_w is

$$\mathrm{ch}\Delta_w = e(w\cdot 0)\mathrm{chDist}(U^-_1) = e(w\cdot 0-(p-1)(\rho+w\rho))\mathrm{chDist}({}^wU^+_1).$$

As $\mathrm{wt}(1\otimes\prod_{\alpha\in R^+_w} x^{p-1}_\alpha) = w\cdot 0-(p-1)(\rho+w\rho)$, $1\otimes\prod_{\alpha\in R^+_w}x^{p-1}_\alpha$ is stabilized by $\mathrm{Dist}({}^wB_1)$, and hence there is a homomorphism of G_1T-modules

$$\psi:\mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}({}^wB_1)}(w\cdot 0-(p-1)(\rho+w\rho))\to\Delta_w$$

such that $1\otimes 1\mapsto 1\otimes\prod_{\alpha\in R^+_w}x^{p-1}_\alpha$. In turn, ψ induces a homomorphism of ${}^wU^+_1$-modules

$$\mathrm{Dist}({}^wU^+_1)\to\Delta_w \quad\text{such that}\quad \mu\mapsto\mu\cdot(1\otimes\prod_{\alpha\in R^+_w}x^{p-1}_\alpha).$$

Writing $\mathrm{Dist}({}^wU^+_1)\simeq\otimes_{\beta\in w(R^+)}\mathrm{Dist}(U_{\beta,1}) = \otimes_{\beta\in w(R^+)}(\coprod_{i=0}^{p-1}\Bbbk\mu^{(i)}_\beta)$, the ${}^wU^+_1$-socle of $\mathrm{Dist}({}^wU^+_1)$ is $\Bbbk(\otimes_{\beta\in w(R^+)}\mu^{(p-1)}_\beta)$. As

$$(\otimes_{\beta\in w(R^+)}\mu^{(p-1)}_\beta)\cdot(1\otimes\prod_{\alpha\in R^+_w}x^{p-1}_\alpha) = (\prod_{\alpha\in R^-_w}\partial^{(p-1)}_\alpha)\otimes 1\neq 0$$

by (2.2) and (4), it follows that ψ is injective, and hence bijective by dimension.

(ii) By (4) one also has a surjective homomorphism of $\bar{\mathcal{D}}(\Omega_w)$-modules

$$\bar{\mathcal{D}}(\Omega_w)/\sum_{\beta\in w(R^+)}\bar{\mathcal{D}}(\Omega_w)x_\beta\to\Delta_w \quad\text{via}\quad \delta\mapsto\delta\cdot(1\otimes\prod_{\alpha\in R^+_w}x^{p-1}_\alpha),$$

which is bijective by dimension. On the other hand, the simplicity of $\bar{\mathcal{D}}(\Omega_w)/\sum_{\beta\in w(R^+)}\bar{\mathcal{D}}(\Omega_w)x_\beta$ follows from the equality

$$(\prod_{\beta\in w(R^+)}x^{p-1}_\beta)(\prod_{\beta\in w(R^+)}\partial^{p-1}_\beta) = 1 \mod \sum_{\beta\in w(R^+)}\bar{\mathcal{D}}(\Omega_w)x_\beta.$$

(iii) If $y \in W \setminus \{w\}$,

$$\begin{aligned}\Gamma(\Omega_y, \bar{\int}_{j_w \circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) &= \Gamma(\Omega_y \cap \Omega_w, \bar{\int}_{i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) \\ &= \Gamma(\Omega_y \cap \Omega_w, \bar{\int}_{i_w|_{\Omega_y \cap \mathcal{B}_w}} (k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) \mid_{\Omega_y \cap \mathcal{B}_w}) \\ &= 0 \quad \text{as } wB \notin \Omega_y.\end{aligned}$$

It now follows that $\bar{\int}_{j_w \circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}$ is coherent over $\bar{\mathcal{D}}$. Finally, the annihilator in $\Bbbk[\Omega_w] \simeq \Bbbk[{}^wU^+]$ of Δ_w contains all x_β^p, $\beta \in w(R^+)$, by (4). It follows that the support of Δ_w on ${}^wU^+$ consists just of the identity element, hence $\mathrm{supp}(\bar{\int}_{j_w \circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}) = \{wB\}$.

(2.4) Using the Bezrukavnikov-Mirkovic-Rumynin derived equivalence [BMR, 3.2], we obtain

Corollary. *Let $w \in W$.*

(i) $\bar{\int}_{j_w \circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}$ *is simple as $\bar{\mathcal{D}}$-module.*

(ii) *If $p > 2(h-1)$, h the Coxeter number of G, and if $D = \Gamma(\mathcal{B}, \mathcal{D})$, there is an isomorphism in the bounded derived category of coherent $\mathcal{D}$-modules*

$$\mathcal{D} \otimes_D^L \Delta_w \simeq \bar{\int}_{j_w \circ i_w} k_{w*}\mathcal{O}_{\mathrm{FN}(w)}.$$

(2.5) **Remark.** For $r \geq 2$ let $\mathrm{FN}^r(w)$ be the r-th Frobenius neighbourhood of wB in $\mathcal{B}_w$ defined by $(a^{p^r} \mid a \in \mathcal{I}_w)$, $\Delta_{w,r} = \Gamma(\mathcal{B}, \int_{j_w \circ i_{w,r}} k_{w*}\mathcal{O}_{\mathrm{FN}^r(w)})$, and G_r (resp. ${}^wB_r$) the r-th Frobenius kernel of G (resp. ${}^wB = wBw^{-1}$). The same arguments as in (2.3) yield

(i) There is an isomorphism of G_rT-modules

$$\Delta_{w,r} \simeq \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}({}^wB_r)} (w \cdot 0 - (p^r - 1)(\rho + w\rho)).$$

(ii) If $D_r(\Omega_w) = \Gamma(\Omega_w, \mathcal{D}_{\mathcal{B},r})$, $\Delta_{w,r}$ is simple as $D_r(\Omega_w)$-module.

(iii) $\int_{j_w \circ i_{w,r}} k_{w*}\mathcal{O}_{\mathrm{FN}^r(w)}$ is simple over $\mathcal{D}_{\mathcal{B},r}$ with $\mathrm{supp}(\int_{j_w \circ i_{w,r}} k_{w*}\mathcal{O}_{\mathrm{FN}^r(w)}) = \{wB\}$.

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