STABILITY AND RIGIDITY OF SPECIAL LAGRANGIAN CONES OVER CERTAIN MINIMAL LEGENDRIAN ORBITS

YOSHIHIRO OHNITA

Dedicated to Professor Katsuei Kenmotsu on his retirement from Tohoku University

ABSTRACT. Special Lagrangian cones in complex Euclidean spaces are obtained as cones over compact minimal Legendrian submanifolds in the odd dimenisonal standard hypersphere. The notion of the stability, the Legendrian stability and the rigidity of special Lagrangian cones were recently introduced and investigated by D. Joyce, M. Haskins etc. In this paper we determine explicitly the stability-index, the Legendrian-index, and the rigidity of special Lagrangian cones over compact irreducible symmeric spaces of type A obtained as minimal Legendrian orbits and over a minimal Legendrian $SU(2)$-orbit. We obtain the examples of stable and rigid special Lagrangian cones in higher dimensions. Moreover we discuss a relationship of these properties with the Hamiltonian stability of minimal Lagrangian submanifolds in complex projective spaces.

INTRODUCTION

A special Lagrangian submanifold in a Ricci-flat Kähler manifold, a so-called Calabi-Yau manifold, has two aspects of a *Lagrangian submanifold* in symplectic geometry and a *calibrated submanifold* in Riemannian geometry. A calibrated submanifold is a minimal submanifold in the sense that the mean curvature vector field vanishes, and more strongly it is a real homologically volume minimizing submanifold.

Recently D. Joyce provided the profound theory on special Lagrangian submanifolds with isolated conical singularities in (almost) Calabi-Yau manifolds and thier deformations, moduli spaces in a series of his papers. His work emphasizes so much the importance of investigation of special Lagrangian cones in complex Euclidean spaces.

The notion of the stability-index, the stability and the rigidity of special Lagrangian cones were introduced by D. Joyce. They are closely related to the deformation of special Lagrangian submanifolds with isolated conical singularities and the regularity of special Lagrangian integral currents. A *special Lagrangian cone* is obtained as a cone over a compact *minimal Legendrian submanifold* in the odd dimensional standard sphere. By the Hopf fibration

2000 *Mathematics Subject Classification.* Primary: 53C40; Secondary: 53C38, 53C42.

Key words and phrases. special Lagrangian submanifold, special Lagrangian cone, minimal submanifold, symmetric space, Legendrian submanifold.

a minimal Legendrian submanifold can be locally projected to a *minimal Lagrangian submanifold* in the complex projective space.

The most fundamental and typical examples are special Lagrangian cones C^m_{HL} over minimal Legendrian orbits of the maximal torus T^{m-1} of the special unitary group $SU(m)$ given by Harvey and Lawson ([8]). M. Haskins showed that a stable special Lagrangian cone in $\mathbf{C}^3$ over a compact minimal Legendrian surface of genus 1 in S^5 is only C^3_{HL} ([7]). The further research on stable special Lagrangian cones in higher dimensions and the stability-index of higher dimensional homogeneous examples are suggested in the paper [7, p62].

Now we assume that Σ is one of compact irreducible symmetric spaces standardly embedded in the odd dimensional standard sphere $S^{2m-1}(1)$ as minimal Legendrian submanifolds in the standard way (see Section 2) :

$$\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)\ (p \geq 3),\ \text{or } E_6/F_4.$$

Note that the rank of these symmetric spaces is equal to $p-1$ and the rank of E_6/F_4 is equal to 2. Let $C\Sigma$ be the special Lagrangian cone in $\mathbf{C}^m$ over Σ. Then we shall show the following.

Theorem. (1) *$C\Sigma$ are all RIGID.*

(2) *If $\Sigma = SU(3), SU(3)/SO(3), SU(6)/Sp(3)\ (p=3), E_6/F_4$, then $C\Sigma$ is STABLE, and hence Legendrian stable.*

(3) *If $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p), p \geq 4$, $C\Sigma$ is NOT stable, in fact NOT Legendrian stable.*

The properties of these minimal Legendrian submanifolds will be discussed in detail and their stability-indices will be determined explicitly. In the last section of this paper we shall discuss such properties of a special Lagrangian cone over a minimal Legendrian $SU(2)$-orbit in $\mathbf{C}^4$.

The results in this paper were partially announced in [17]. In November 2004, Mark Haskins has visited Kyushu University and Tokyo Metropolitan University. The author could have nice discussion with him about this subject there. The author would like to thank Mark Haskins for his valuable suggestion of a problem on the existence of stable special Lagrangian cones in higher dimensions.

1. Special Lagrangian cones and their stability-indices

In this section we shall describe some fundamental definitions and properties which are necessary in the later sections (cf. [6],[7], [10],[11],[12]).

1.1. **Special Lagrangian submanifolds of Calabi-Yau manifolds.** In complex Euclidean slpace $\mathbf{C}^m \cong \mathbf{R}^{2m}$, we recall the notion of special Lagrangian submanifolds. The natural group action of $SU(m) \subset U(m)$ preserves the standard Kähler form (symplectic form) defined as

$$\omega := \sqrt{-1}\sum_{i=1}^{m} dz^i \wedge d\bar{z}^i$$

and the standard complex volume form defined by

$$\Omega := dz^1 \wedge \cdots \wedge dz^m.$$

We decompose Ω into real and imaginary parts by

$$\Omega = \mathrm{Re}(\Omega) + \sqrt{-1}\mathrm{Im}(\Omega).$$

Then $\mathrm{Re}(\Omega)$ and $\mathrm{Im}(\Omega)$ are parallel real n-forms on $\mathbf{C}^m$.

The calibrated submanifolds by $\mathrm{Re}(\Omega)$ are characterized by the condition that the restrictions of ω and $\mathrm{Im}(\Omega)$ to the submanifold vanish. The *special Lagrangian submanifold* in $\mathbf{C}^m$ is defined as such a submanifold Harvey and Lawson showed that a minimal Lagrangian submanifold in $\mathbf{C}^m$ is a special Lagrangian submanifold.

In general, suppose that (M, g) is a Riemannian manifold with holonomy group contained in $SU(m)$, and such a Riemannian manifold becomes a *Calabi-Yau* Kähler manifold of complex dimension m. Then the parallel Kähler form ω and the parallel complex volume form Ω are defined on the whole M, and $\mathrm{Re}(\Omega)$ defines a calibration on M. The calibrated submanifolds with respect to $\mathrm{Re}(\Omega)$ are characterized by the condition that the pull-backs of ω and $\mathrm{Im}(\Omega)$ to the submanifold vanish. An m-dimensional submanifold X in a Calabi-Yau manifold is called a *special Lagrangian submanifold* if the pull-backs of both ω and $\mathrm{Im}(\Omega)$ to X vanish.

For each constant $\theta \in \mathbf{R}$, we also can consider a calibration defined by $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ and its corresponding calibrated submanifolds. We also call such a calibrated submanifold a *special Lagrangian submanifolds* (with respect to $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$) if the pull-backs of both ω and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)$ to X vanish. Let X be a Lagrangian submanifold immersed in a Calabi-Yau manifold M. Then we know that X is a minimal submanifold in M if and only if X is a special Lagrangrian submanifold with respect to the calibration $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ for some $\theta \in \mathbf{R}$.

1.2. **Special Lagrangian cones.** Let $S^{2m-1}(1)$ denote the unit standard hypersphere of $\mathbf{C}^m$. Let Σ be an $(m-1)$-dimensional smooth submanifold immersed in $S^{2m-1}(1)$ defined by an immersion $\varphi : \Sigma \longrightarrow S^{2m-1}(1)$. The *cone* $C = C\Sigma$ over Σ in $\mathbf{C}^m$ is defined by an immersion

$$\Phi : \Sigma \times [0, \infty) \ni (\sigma, t) \longmapsto t\varphi(\sigma) \in \mathbf{C}^m.$$

Then C has an isolated singularity at the origin 0 and $C' := C \setminus \{0\}$ is an m-dimensional smooth submanifold immersed in $\mathbf{C}^m$ defined by the immersion

$$\Phi' : \Sigma \times (0, \infty) \ni (\sigma, t) \longmapsto t\varphi(\sigma) \in \mathbf{C}^m.$$

Let $\pi : S^{2m-1}(1) \longrightarrow \mathbf{C}P^{m-1}$ be the Hopf fibration, which is a Riemannian submersion onto the $(m-1)$-dimensional complex projective space $\mathbf{C}P^{m-1}$ of constant holomorphic sectional curvature 4. Then $C\Sigma$ is a Lagrangian cone with an isolated singularity at 0 if and only if Σ is a Legendrian submanifold in $S^{2m-1}(1)$ with the standard contact structure, and then the immersion $\pi \circ \varphi : \Sigma \longrightarrow S^{2m-1}(1)$ defines a Lagrangian submanifold immersed in $\mathbf{C}P^{m-1}$.

Moreover since the mean curvature vectors of these submanifolds correspond each other, we know the following fundamental fact (cf. [7]).

Proposition 1.1. *The following three conditions on local properties of these submanifolds are equivalent each other:*

(a) $C\Sigma$ *is a special Lagrangian cone in* $\mathbf{C}^m$.

(b) Σ *is a minimal Legendrian submanifold in* $S^{2m-1}(1)$ *with respect to its standard contact structure,*

(c) $\pi(\Sigma)$ *is a minimal Lagrangian submanifold in* $\mathbf{C}P^m$.

Example 1.1. In Harvey-Lawson [8] the following example of a special Lagrangian cone in $\mathbf{C}^m$ was given as

$$C^m_{HL} := \{(z_1, \cdots, z_m) \in \mathbf{C}^m \mid (\sqrt{-1})^{m+1} z_1 \cdots z_m \in \mathbf{R}, |z_1| = \cdots = |z_m|\}.$$

Then

$$\Sigma^{m-1}_{HL} := C^m_{HL} \cap S^{2m-1}(1) \subset S^{2m-1}(1)$$

is a minimal Legendrian orbit of the maximal torus of $SU(m)$, which is isometric to an $(m-1)$-dimensional flat torus T^{m-1}.

Let Δ and Δ_Σ be the Laplacians of (C', g) and (Σ, g_Σ) on functions, respectively. A function u on C' is called a *homogeneous function of order* α on C' if u satisfies $u \circ t = t^\alpha u$ for each $t > 0$. Then such a function can be expressed as $u(r\sigma) = r^\alpha v(\sigma)$ for some function v on Σ. The relationship between Δ and Δ_Σ is given by the formula

$$\Delta u(r\sigma) = r^{\alpha-2}(\Delta_\Sigma v(\sigma) - \alpha(\alpha + m - 2)v(\sigma)). \tag{1.1}$$

Hence we see that u is harmonic if and only if v is an eigenfunction on Σ with eigenvalue $\alpha(\alpha + m - 2)$.

Assume that $m > 2$. Set

$$\mathcal{D}_\Sigma := \{\alpha \in \mathbf{R} \mid \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\},$$

which is a countable and discrete subset of $\mathbf{R}$. For each $\alpha \in \mathcal{D}_\Sigma$, we denote by $m_\Sigma(\alpha)$ the multiplicity for eigenvalue $\alpha(\alpha+m-2)$ of Δ_Σ, which is equal to the dimension of vector space of all homogeneous harmonic functions of order α on C'. Then we define a monotone increasing, upper semi-continuous function $N_\Sigma : \mathbf{R} \longrightarrow \mathbf{Z}$ as

$$N_\Sigma(\delta) := - \sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha)$$

if $\delta < 0$ and

$$N_\Sigma(\delta) := \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta]} m_\Sigma(\alpha)$$

if $\delta \geqq 0$.

Definition 1.1. The *stability-index* of a special Lagrangian cone C is defined by

$$\text{s-ind}(C) := N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G_\Sigma, \tag{1.2}$$

where $b^0(\Sigma)$ denotes the 0-th Betti number of Σ, i.e. the number of connected components of Σ and G_Σ denotes a maximal compact subgroup of $SU(m)$ preserving the special Lagrangian cone C, or equivalently the minimal Legendrian submanifold Σ.

Note that $m_\Sigma(0) = b^0(\Sigma)$, $m_\Sigma(1) \geq 2m$ if Σ is not totally geodesic, $m_\Sigma(2) \geq m^2 - 1 - \dim G_\Sigma$. Since $N_\Sigma(2) \geq m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, we have s-ind$(C) \geq 0$ if Σ is not totally geodesic. If Σ is totally geodesic, then s-ind$(C) = -m$.

A special Lagrangian cone C is called *stable* if s-ind$(C) = 0$. A special Lagrangian cone C is called *rigid* if $m_\Sigma(2) = m^2 - 1 - \dim G_\Sigma$. We see that a special Lagrangian cone C is stable if and only if the following three conditions are satisfied

(1) $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$,
(2) $m_\Sigma(1) = 2m$,
(3) $m_\Sigma(2) = m^2 - 1 - \dim G_\Sigma$.

The *Legendrian-index* of a special Lagrangian cone C ([7]) is defined by

$$\text{l-ind}(C) := \sum_{\alpha \in \mathcal{D}_\Sigma \cap (0,2)} m_\Sigma(\alpha). \tag{1.3}$$

A special Lagrangian cone C is *Legendrian-stable* ([7]) if l-ind$(C) = 2m$. A special Lagrangian cone C is Legendrian-stable if and only if $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$ and $m_\Sigma(1) = 2m$. By the definitions C is stable if and only if C is rigid and Legendrian-stable.

Here we shall mention a relationship of the stability of special Lagrangian cones with the Hamiltonian stability of minimal Lagrangian submanifolds in complex projective spaces (cf. [1]).

Assume that $\psi : L \longrightarrow \mathbf{C}P^{m-1}$ is a minimal Lagrangian immersion of an $(m-1)$-dimensional connected compact smooth manifold L into a complex projective space. Since the pull-back S^1-bundle $\psi^{-1}\pi : \psi^{-1}S^{2m-1}(1) \to L$ is flat, there is a connected integral manifold Σ of the horizintal distribution on $\psi^{-1}S^{2m-1}(1)$, and hence it gives a minimal Legendrian immersion $\varphi : \Sigma \to S^{2m-1}(1)$ and a covering map $\psi^{-1}\pi : \Sigma \to L$. We denote by $\rho : \pi_1(L) \to S^1$ the holonomy homomorphism of the flat S^1-bundle $\psi^{-1}S^{2m-1}(1)$ over L. Then the following holds.

Proposition 1.2. *Suppose that ρ is nontrivial. If the special Lagrangian cone $C\Sigma$ over Σ in $\mathbf{C}^m$ is stable, then a minimal Lagrangian submanifold L in $\mathbf{C}P^{m-1}$ is Hamiltonian stable.*

Proof. We may assume that φ is not totally geodesic. For each $v \in \mathbf{C}^m$, we define a smooth function f_v on Σ by

$$(f_v)(x) := \langle \varphi(x), v \rangle \quad (x \in \Sigma).$$

Let $\rho : \pi_1(\Sigma) \longrightarrow S^1$ be the holonomy homomorphism of the pull-back S^1-bundle from the Hopf S^1-bundle $\pi : S^{2m-1}(1) \longrightarrow \mathbf{C}P^{m-1}$ by the Lagrangian immersion ψ. Here S^1 is considered as the center of the unitary group $U(m)$.

Set $\Gamma := \rho(\pi_1(\Sigma))$, which is a finite subgroup of S^1. Let Γ be the deck transformation group of the covering map $\psi^{-1}\pi : \Sigma \longrightarrow L$. Suppose that there is a vector $v \in \mathbf{C}^m$ such that $f_v(xc) = f_v(x)$ for each $c \in \Gamma$ and each $x \in \Sigma$. Since $\varphi(xc) = \varphi(x)\rho(c)$, we have $\langle \varphi(x)a, v\rangle = \langle \varphi(x), v\rangle$ for each $a \in \rho(\Gamma)$ and each $x \in \Sigma$. By the non-triviality of Γ, there is $a \in \Gamma$ with $a \neq 1$. Since $\langle \varphi(x), va^{-1} - v\rangle = 0$ for all $x \in \Sigma$, by the fullness of φ we have $va^{-1} = v$. As $a \neq 1$, v must be zero and thus $f_v = 0$. Hence by the assumption on the stability we conclude that Σ has no nonzero eigenvalue smaller than $2m$. Therefore Σ is Hamiltonian stable. $\square$

It can happen that L becomes Hamiltonian stable even if $C\Sigma$ is not stable. Such examples will be shown in the later sections.

1.3. **Special Lagrangian submanifolds with isolated conical singularities.** Here we mention the results of Joyce on the deformation of a compact special Lagrangian submanifold X with isolated conical singularities or the local structure of moduli spaces around X, and the regularity of special Lagrangian varieties, which are described in terms of the stability-index and the rigidity of special Lagrangian cones.

Let $\mathcal{M}$ be the moduli space of compact special Lagrangian submanifolds with isolated conical singularities embedded in M. McLean [14] showed that if $X \in \mathcal{M}$ is smooth (i.e. without singularities), then the moduli space $\mathcal{M}$ is a smooth manifold of dimension $b^1(X)$ around X.

Joyce [11] showed that if X is a special Lagrangian submanifold with isolated conical singularities $C_1, \cdots, C_k$, then the dimension of the obstruction space $\mathcal{O}_X$ of X is equal to the sum of stability-indices of special Lagrangian cones $C_1, \cdots, C_k$:

$$\dim \mathcal{O}_X = \sum_{i=1}^{k} \text{s-ind}(C_i).$$

This means that s-ind(C) of a special Lagrangian cone C is the dimension of the obstruction space to deforming a special Lagrangian submanifold X in a Calabi-Yau manifold with a conical singularity with cone C, and that if C is *stable* then the deformation theory of X simplifies.

That a special Lagrangian cone C is *rigid* means that if all infinitesimal deformations of C as a special Lagrangian cone comes from rotations of C by $SU(m)$. Next we mention the Joyce's regularity results of special Lagrangian integral currents, or special Lagrangian varieties. Geometric measure theory implies the compactness of the space of such objects. Suppose that X is a special Lagrangian integral current and has the multiplicity 1 tangent cone at $x \in \mathrm{supp}X$. Joyce showed that if the tangent cone of X at x is a rigid special Lagrangian cone, then X has an isolated conical singularity at x.

So it is actually interesting and important to investigate explicitly the stability and rigidity of special Lagrangian cones.

Joyce and Marshall proved that C^3_{HL} is stable and C^m_{HL} is unstable if $m \geqq 4$, and C^m_{HL} is rigid if and only if $m \neq 8, 9$, and they determined their stability-indices and Legendrian-indices explicitly (cf. [11]). By the spectral analysis on surfaces Haskins showed that a stable special Lagrangian cone in $\mathbf{C}^3$ over a minimal Legendrian torus in S^5 is only C^3_{HL} ([7]).

Problem. Construct and classify stable special Lagrangian cones in complex Euclidean spaces.

2. Stability-index of special Lagrangian cones over certain compact irreducible symmetric spaces

In this section we shall discuss a class of special Lagrangian cones constructed by the Lie theoretic method including the Harvey-Lawson cones C^m_{HL}. Let (U, G) be an Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Set $\dim(U/G) = 2m$. Let $\langle\ ,\ \rangle_{\mathfrak{u}}$ denote the $\mathrm{Ad}(U)$-invariant inner product of $\mathfrak{u}$ defined by (-1)-times Killing-Cartan form of $\mathfrak{u}$. We decompose $\mathfrak{g}$ into the direct sum of the semisimple part $\mathfrak{g}_{ss}$ and the center $\mathfrak{c}(\mathfrak{g})$ as follows : $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{c}(\mathfrak{g})$. There is an element $Z \in \mathfrak{c}(\mathfrak{g})$ such that $\mathrm{ad}Z$ defines the invariant complex structure of (U, G). Relative to the complex structure the subspace $\mathfrak{p}$ can be identified with a complex Euclidean space $\mathbf{C}^m$. We take the decomposition of (U, G) into irreducible Hermitian symmetric pairs of compact type :

$$(U, G) = (U_1, G_1) \oplus \cdots \oplus (U_s, G_s). \tag{2.1}$$

Set $\dim(U_i/G_i) = 2m_i$ for $i = 1, \cdots, s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i, G_i) for each $i = 1, 2, \cdots, s$. Assume that there is an element $\eta_i \in \mathfrak{p}_i$ satisfying the condition $(\mathrm{ad}\eta_i)^3 + 4(\mathrm{ad}\eta_i) = 0$. Choose positive numbers $c_1 > 0, \cdots, c_s > 0$ with $\sum_{i=1}^s 1/c_i = 1/c$. Put $a_i = 1/\sqrt{2c_i m_i}$ for each $i = 1, \cdots, s$. Set $\hat{L}_i = \mathrm{Ad}(G_i)(a_i\eta_i) \subset S^{2m_i-1}(c_i/4) \subset \mathfrak{p}_i$, which is an irreducible symmetric R-space standard embedded in a complex Euclidean space $\mathfrak{p}_i$.

Set $\eta = a_1\eta_1 + \cdots + a_s\eta_s \in \mathfrak{p}$. Set $\hat{L} = \mathrm{Ad}(G)(\eta) \subset S^{2m-1}(c/4) \subset \mathfrak{p}$, which is a symmetric R-space standard embedded in a complex Euclidean space $\mathfrak{p} \cong \mathbf{C}^m$. Note that we have the inclusions

$$\hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s \subset S^{2m_1-1}(c_1/4) \times \cdots \times S^{2m_s-1}(c_s/4) \subset S^{2m-1}(c/4). \tag{2.2}$$

Note that $\hat{L}$ is a compact H-minimal Lagrangian submanifold embedded in $\mathbf{C}^m$ (see [3]).

We take an orthogonal decomposition $\mathfrak{c}(\mathfrak{g}) = \mathfrak{c}^0 \oplus \{Z\}_{\mathbf{R}}$ of $\mathfrak{c}(\mathfrak{g})$. Let $\mathfrak{g}^0 := \mathfrak{g}_{ss} \oplus \mathfrak{c}^0$ and G^0 denote the analytic subgroup of G generated by $\mathfrak{g}^0$. Set $\Sigma = \mathrm{Ad}(G^0)(\eta) \cong G^0/K^0 \subset S^{2m-1}(c/4) \subset \mathfrak{p}$, where $K^0 = \{a \in G^0 \mid \mathrm{Ad}(a)(\eta) = \eta\}$. Then Σ is a Legendrian submanifold in $S^{2m-1}(c/4)$. Moreover Σ is a minimal submanifold in $S^{2m-1}(c/4)$ if and only if $c_i m_i = cm$ for each $i = 1, 2, \cdots, s$. Thus we obtain

Proposition 2.1. *$C\Sigma$ is a special Lagrangian cone in $\mathbf{C}^m$ if and only if the condition $c_i m_i = cm$ is satisfied for each $i = 1, 2, \cdots, s$.*

In the case when $(U_i, G_i) = (SU(2), S(U(1) \times U(1)))$ for all i, the above special Lagrangian cone $C\Sigma$ coincides with the Harvey-Lawson's special Lagrangian cone C^m_{HL}.

In the case when (U, G) is irreducible, i.e. $s = 1$, from the classification theory of symmetric R-spaces, Σ is one of symmetric spaces of compact type in the following list :

(a) S^{m-1}.
(b) $SU(p)$, $m = p^2$.
(c) $SU(p)/SO(p)$, $m = \frac{(p-1)(p+2)}{2} + 1$.
(d) $SU(2p)/Sp(p)$, $m = (p-1)(2p+1) + 1$.
(e) E_6/F_4, $m = 27$.

Here $p \geq 3$. Note that they are connected, simply connected and compact irreducible symmetric spaces whose restricted root systems are of type A, and the rank of the symmetric spaces is equal to $p - 1$ and the rank of E_6/F_4 is 2. They are the standard embeddings by the first eigenfunctions of the Laplacian (cf. [16]).

Suppose that Σ is a compact embedded minimal Legendrian submanifold of $S^{2m-1}(1)$ given by the standard embedding of the above symmetric spaces of compact type. Let $C\Sigma$ be a special Lagrangian cone over Σ in $\mathbf{C}^m$. Then we shall show

Theorem 2.1. (1) *They all $C\Sigma$ are rigid.*

(2) *If $\Sigma = SU(3), SU(3)/SO(3), SU(6)/Sp(3) (p = 3), E_6/F_4$, then $C\Sigma$ is stable and thus Legendrian stable.*

(3) *If $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)$, $p \geq 4$ then $C\Sigma$ is not Legendrian stable and thus not stable.*

Remark. In case (a), $\Sigma = S^{m-1}$ is a totally geodesic Legendrian submanifold embedded in $S^{2m-1}(1)$ and thus $C\Sigma$ is a Lagrangian vector subspace of $\mathbf{C}^m$.

In order to determine the stability-indices of special Lagranian cones over these minimal Legendrian submanifolds $\Sigma = G^0/K^0$, we shall examine explicitly the eigenvalues and their multiplicities of the Laplacian of compact irreducible symmetric spaces G^0/K^0 by the theory of spherical functions on compact symmetric spaces (cf. [19]). In the calculation we use the results described in [1].

First we prepare a useful algebraic lemma for our calculation. Let $(m_1, \cdots, m_p)$ be a p-tuple of real numbers satisfying the conditions

$$(2.3) \qquad \sum_{i=1}^{p} m_i = 0 \quad \text{and} \quad 0 \leq m_i - m_{i+1} \in \mathbf{Z} \text{ for each } i = 1, 2, \cdots, p-1.$$

Then note that $m_i \in (1/p)\mathbf{Z}$ for each $i = 1, 2, \cdots, p-1$. In fact, if we set $\mathbf{Z} \ni k_i := m_i - m_{i+1} \geq 0$, then we have

$$m_p = -\frac{1}{p}\sum_{j=1}^{p-1} jk_j,$$

$$m_i = k_i + \cdots + k_{p-1} - \frac{1}{p}\sum_{j=1}^{p-1} jk_j \quad (i = 1, 2, \cdots, p-1).$$

Lemma 2.1. *Fix a positive real number $t > 0$. Define a function Q with respect to $m_1, \cdots, m_p$ or $k_1, \cdots, k_{p-1}$ by*

$$Q := \sum_{i=1}^{p}(m_i)^2 - t\sum_{i=1}^{p} im_i. \tag{2.4}$$

(1) *If $(m_1, \cdots, m_p) = (1, 0, \cdots, 0, -1)$ i.e. $(k_1, \cdots, k_{p-1}) = (1, 0, \cdots, 0, 1)$, then Q attains $Q = 2 + t(p-1)$.*

(2) *If $(m_1, \cdots, m_p) = (\frac{p-1}{p}, -\frac{1}{p}, \cdots, -\frac{1}{p})$ i.e. $(k_1, \cdots, k_{p-1}) = (1, 0, \cdots, 0)$, or $(m_1, \cdots, m_p) = (\frac{1}{p}, \cdots, \frac{1}{p}, -\frac{p-1}{p})$ $((k_1, \cdots, k_{p-1}) = (0, \cdots, 0, 1))$, then Q attains*

$$Q = \frac{p-1}{p} + t\frac{p-1}{2} < 2 + t(p-1).$$

(3) *Assume that $p \geq 4$. If $(m_1, \cdots, m_p) = (\frac{p-2}{p}, \frac{p-2}{p}, -\frac{2}{p}, \cdots, -\frac{2}{p})$ i.e. $(k_1, \cdots, k_{p-1}) = (0, 1, 0, \cdots, 0)$ or $(m_1, \cdots, m_p) = (\frac{2}{p}, \cdots, \frac{2}{p}, -\frac{p-2}{p}, -\frac{p-2}{p})$ i.e. $(k_1, \cdots, k_{p-1}) = (0, \cdots, 0, 1, 0)$, then Q attains*

$$\frac{p-1}{p} + t\frac{p-1}{2} < Q = \frac{2(p-2)}{p} + t(p-2) < 2 + t(p-1).$$

(4) *$Q = 2 + t(p-1)$ if and only if $(m_1, \cdots, m_p) = (1, 0, \cdots, 0, -1)$ i.e. $(k_1, \cdots, k_{p-1}) = (1, 0, \cdots, 0, 1)$.*

(5) *$Q < 2 + t(p-1)$ if and only if $(m_1, \cdots, m_p)$ or $(k_1, \cdots, k_{p-1})$ is one of the following table :*

p	$(k_1, \cdots, k_{p-1})$	$(m_1, \cdots, m_p)$	Q
≥ 3	$(1, 0, \cdots, 0, 0)$	$(\frac{p-1}{p}, -\frac{1}{p}, \cdots, -\frac{1}{p})$	$\frac{p-1}{p} + t\frac{p-1}{2}$
≥ 3	$(0, 0, \cdots, 0, 1)$	$(\frac{1}{p}, \cdots, \frac{1}{p}, -\frac{p-1}{p})$	$\frac{p-1}{p} + t\frac{p-1}{2}$
4	$(0, 1, 0)$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$1 + 2t$
≥ 5	$(0, 1, 0, \cdots, 0)$	$(\frac{p-2}{p}, \frac{p-2}{p}, -\frac{2}{p}, \cdots, -\frac{2}{p})$	$\frac{2(p-2)}{p} + t(p-2)$
≥ 5	$(0, \cdots, 0, 1, 0)$	$(\frac{2}{p}, \cdots, \frac{2}{p}, -\frac{p-2}{p}, -\frac{p-2}{p})$	$\frac{2(p-2)}{p} + t(p-2)$
6	$(0, 0, 1, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$\frac{3}{2} + t\frac{9}{2}$
7	$(0, 0, 1, 0, 0, 0)$	$(\frac{4}{7}, \frac{4}{7}, \frac{4}{7}, -\frac{3}{7}, -\frac{3}{7}, -\frac{3}{7}, -\frac{3}{7})$	$\frac{12}{7} + 6t$
7	$(0, 0, 0, 1, 0, 0)$	$(\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, -\frac{4}{7}, -\frac{4}{7}, -\frac{4}{7})$	$\frac{12}{7} + 6t$

Proof. The statements (1), (2) and (3) are obtained by direct computations. The function Q can be described in terms of $k_1, \cdots, k_{p-1}$ as the formula :

$$Q = \sum_{i=1}^{p-1} \{ \sum_{j=1}^{i} (1 - \frac{i}{p}) j k_j + \sum_{j=i+1}^{p-1} i (1 - \frac{j}{p}) k_j \} k_i + t \sum_{i=1}^{p-1} \frac{i(p-i)}{2} k_i. \tag{2.5}$$

The statements (4),(5) follow from this formula. $\square$

The case $\Sigma = (SU(p) \times SU(p))/SU(p)$: In this case note that $m - 1 = \dim \Sigma = p^2 - 1$, $m = p^2$, $2m = 2p^2$ and $m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim (SU(p) \times SU(p)) = (p^2 - 1)^2$.

Let $\{\varepsilon_1, \cdots, \varepsilon_p\}$ be the standard orthonormal basis of a p-dimensional Euclidean vector space $\mathbf{R}^p$. Set

$$\begin{aligned} & D(SU(p)) \\ =& \Big\{ \sum_{i=1}^{p} m_i \varepsilon_i \mid \sum_{i=1}^{p} m_i = 0, 0 \leq m_i - m_{i+1} \in \mathbf{Z} \ (i = 1, 2, \ldots, p-1) \Big\} \\ =& \Big\{ \sum_{i=1}^{p-1} k_i \Lambda_i \mid 0 \leq k_i \in \mathbf{Z} \ (i = 1, 2, \ldots, p-1) \Big\}. \end{aligned} \tag{2.6}$$

Here $k_i = m_i - m_{i+1}$ $(i = 1, \cdots, p-1)$ and $\{\Lambda_1, \cdots, \Lambda_{p-1}\}$ is the fundamental weight system of $SU(p)$ defined by

$$\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{p} \sum_{j=1}^{p} \varepsilon_j \quad (i = 1, 2, \cdots, p-1).$$

We know that there is a bijective correspondence between $D(SU(p))$ and the complete set of all inequivalent complex irreducible representations of $SU(p)$. Then for each $\Lambda = \sum_{i=1}^{p} m_i \varepsilon_i \in D(SU(p))$ the eigenvalue a_Λ of the Casimir operator on a complex irreducible representation with highest weight Λ is equal to

$$-a_\Lambda = \sum_{i=1}^{p} (m_i)^2 - 2 \sum_{i=1}^{p} i m_i \tag{2.7}$$

and the corresponding eigenvalue of Δ_Σ is given by

$$\lambda = (-a_\Lambda) \frac{1}{2p} \cdot 2C^{-1} = (-a_\Lambda) \frac{1}{2p} \cdot 2 \cdot p^2 = (-a_\Lambda) p = pQ \tag{2.8}$$

because of $C = \frac{4}{p^2 c} = \frac{1}{p^2}$ by [1, p594]. Here Q is a function defined in Lemma 2.1. For each $\Lambda \in D(SU(p))$, we denote by d_Λ the dimension of a complex irreducible representation with highest weight Λ. The dimension d_Λ is given by the Weyl's dimension formula. The multiplicity $m(\lambda)$, i.e. the dimension of the eigenspace, for the eigenvalue λ of the Laplacian Δ_Σ is equal to

$$m(\lambda) = \sum_{\Lambda \in D(SU(p)), \lambda = (-a_\Lambda) p} (d_\Lambda)^2.$$

First we consider the case $p = 3$. Then (2.7) becomes

$$-a_\Lambda = \frac{2}{3}(k_1^2 + k_1k_2 + k_2^2) + 2(k_1 + k_2). \tag{2.9}$$

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, then $(-a_\Lambda) \cdot 3 = \frac{8}{3}3 = 8$ and $d_\Lambda = 3$.
(2) If $(k_1, k_2) = (1, 1)$, then $(-a_\Lambda) \cdot 3 = 6 \cdot 3 = 18$ and $d_\Lambda = 8$.
(3) If $(k_1, k_2) =$ otherwise , then $(-a_\Lambda) \cdot 3 \geq 20 > 18$.

Thus all eigenvalues λ and their multiplicitity $m(\lambda)$ of Δ_Σ between 0 and $2m = 18$ are determined as follows :

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
8	1	18
18	2	64

Hence we have $N_\Sigma(2) = m_\Sigma(0)+m_\Sigma(1)+m_\Sigma(2)$, $m_\Sigma(0) = 1 = b^0(\Sigma)$, $m_\Sigma(1) = 18 = 2m$, and $m^2 - 1 - \dim G_\Sigma = 9^2 - 1 - (8 + 8) = 64 = m_\Sigma(2)$. Therefore we conclude that s-ind$(C) = 0$.

Next we treat the case $p \geq 4$. By Lemma 2.1 we obtain the following table of all $\Lambda \in D(SU(p))$ corresponding to eigenvalues $\lambda \leq 2m = 2p^2$:

p	Λ	$(k_1, \cdots, k_{p-1})$	$\lambda = pQ$	d_Λ
≥ 3	$\Lambda_1 + \Lambda_{p-1}$	$(1, 0, \cdots, 0, 1)$	$2p^2$	$p^2 - 1$
≥ 3	Λ_1	$(1, 0, \cdots, 0, 0)$	$p^2 - 1$	p
≥ 3	Λ_{p-1}	$(0, 0, \cdots, 0, 1)$	$p^2 - 1$	p
4	Λ_2	$(0, 1, 0)$	20	6
≥ 5	Λ_2	$(0, 1, 0, \cdots, 0)$	$2(p + 1)(p - 2)$	$\frac{p(p-1)}{2}$
≥ 5	Λ_{p-2}	$(0, \cdots, 0, 1, 0)$	$2(p + 1)(p - 2)$	$\frac{p(p-1)}{2}$
6	Λ_3	$(0, 0, 1, 0, 0)$	63	20
7	Λ_3	$(0, 0, 1, 0, 0, 0)$	96	35
7	Λ_4	$(0, 0, 0, 1, 0, 0)$	96	35

Note that the (nonzero) first eigenvalue of Δ_Σ is $p^2 - 1 = \dim \Sigma$.

By using these results, we determine all $\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$ by $\lambda = \alpha(\alpha+m-2)$, that is, $\alpha = (\sqrt{(m - 2)^2 + 4\lambda} - (m - 2))/2$ as follows :

If $p \geq 8$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
$p^2 - 1$	1	$2p^2$
$2(p + 1)(p - 2)$	$\frac{\sqrt{(p^2-2)^2+8(p+1)(p-2)}-(p^2-2)}{2}$	$\frac{p^2(p-1)^2}{2}$
$2p^2$	2	$(p^2 - 1)^2$

If $p = 7$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
48	1	98
80	$(\sqrt{2529}-47)/2$	882
96	$(\sqrt{2593}-47)/2$	2450
98	2	2304

If $p = 6$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
35	1	72
56	$\sqrt{345}-17$	450
63	$\sqrt{352}-17$	400
72	2	1225

If $p = 5$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
24	1	50
36	$(\sqrt{673}-23)/2$	200
50	2	576

If $p = 4$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
15	1	32
20	$\sqrt{69}-7$	36
32	2	225

We obtain s-ind$(C) > 0$ and thus C is not stable.

The case $\Sigma = SU(p)/SO(p)$: In this case note that $m - 1 = \dim \Sigma = (p-1)(p+2)/2$, $m = p(p+1)/2$, $2m = p(p+1)$ and $m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim SU(p) = p^2(p+3)(p-1)/4$

The subset $D(SU(p), SO(p)) \subset D(SU(p))$ is defined by

$$\begin{aligned} &D(SU(p), SO(p)) \\ =&\{2\sum_{i=1}^{p} m_i\varepsilon_i \mid \sum_{i=1}^{p} m_i = 0, 0 \le m_i - m_{i+1} \in \mathbf{Z}\ (i = 1, 2, \ldots, p-1)\} \\ =&\{\sum_{i=1}^{p-1} k_i M_i \mid 0 \le k_i \in \mathbf{Z}\ (i = 1, 2, \ldots, p-1)\} \end{aligned} \tag{2.10}$$

Here $k_i = m_i - m_{i+1}$ $(i = 1, \cdots, p-1)$ and $\{M_i \mid i = 1, \cdots, p-1\}$ is the fundamental weight system of $(SU(p), SO(p))$ defined by

$$M_i = 2\Lambda_i = 2(\varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{p}\sum_{j=1}^{p} \varepsilon_j) \quad (i = 1, 2, \cdots, p-1).$$

We know that there is a bijective correspondence between $D(SU(p), SO(p))$ and the complete set of all inequivalent spherical representations of the compact symmetric pair $(SU(p), SO(p))$. Then for each $\Lambda = 2\sum_{i=1}^{p} m_i\varepsilon_i \in D(SU(p), SO(p))$ we have

$$-a_\Lambda = 4\sum_{i=1}^{p}(m_i)^2 - 4\sum_{i=1}^{p} i m_i \tag{2.11}$$

and the corresponding eigenvalue of Δ_Σ is given by

$$\lambda = (-a_\Lambda)\frac{1}{2p}C^{-1} = (-a_\Lambda)\frac{1}{2p}\frac{p^2}{2} = (-a_\Lambda)\frac{p}{4} = pQ \tag{2.12}$$

because of $C = \frac{8}{p^2 c} = \frac{2}{p^2}$ by [1, p594]. Here Q is a function defined in Lemma 2.1. The multiplicity $m(\lambda)$, i.e. the dimension of the eigenspace, with eigenvalue λ of the Laplacian Δ_Σ is equal to

$$m(\lambda) = \sum_{\Lambda \in D(SU(p),SO(p)), \lambda = (-a_\Lambda)p/4} d_\Lambda.$$

First we consider the case $p = 3$. Then (2.11) becomes

$$-a_\Lambda = \frac{8}{3}(k_1^2 + k_1k_2 + k_2^2) + 4(k_1 + k_2). \tag{2.13}$$

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, then $(-a_\Lambda) \cdot \frac{3}{4} = \frac{20}{3}\frac{3}{4} = 5$ and $d_\Lambda = 6$.
(2) If $(k_1, k_2) = (1, 1)$, then $(-a_\Lambda) \cdot \frac{3}{4} = 16 \cdot \frac{3}{4} = 12$ and $d_\Lambda = 27$.
(3) If $(k_1, k_2) =$ otherwise , then $(-a_\Lambda) \cdot \frac{3}{4} > 13 > 12$.

Thus all eigenvalues λ and their multiplicities of Δ_Σ between 0 and $2m = 12$ are determined as follows :

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
5	1	12
12	2	27

Hence we have $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, $m_\Sigma(0) = 1 = b^0(\Sigma)$, $m_\Sigma(1) = 12 = 2m$, and $m^2 - 1 - \dim G_\Sigma = 6^2 - 1 - (9-1) = 27 = m_\Sigma(2)$. Therefore we conclude that s-ind$(C) = 0$.

Next we treat the case $p \geq 4$. By Lemma 2.1 we obtain the following table of all $\Lambda \in D(SU(p), SO(p))$ corresponding to eigenvalues $\lambda \leq 2m = p(p+1)$:

p	Λ	$(k_1, \cdots, k_{p-1})$	$\lambda = pQ$	d_Λ
≥ 3	$2\Lambda_1 + 2\Lambda_{p-1}$	$(1, 0, \cdots, 0, 1)$	$p(p+1)$	$\frac{(p-1)p^2(p+3)}{4}$
≥ 3	$2\Lambda_1$	$(1, 0, \cdots, 0, 0)$	$\frac{(p-1)(p+2)}{2}$	$\frac{p(p+1)}{2}$
≥ 3	$2\Lambda_{p-1}$	$(0, 0, \cdots, 0, 1)$	$\frac{(p-1)(p+2)}{2}$	$\frac{p(p+1)}{2}$
4	$2\Lambda_2$	$(0, 1, 0)$	12	20
≥ 5	$2\Lambda_2$	$(0, 1, 0, \cdots, 0)$	$(p-2)(p+2)$	$\frac{p^2(p+1)(p-1)}{12}$
≥ 5	$2\Lambda_{p-2}$	$(0, \cdots, 0, 1, 0)$	$(p-2)(p+2)$	$\frac{p^2(p+1)(p-1)}{12}$
6	$2\Lambda_3$	$(0, 0, 1, 0, 0)$	36	175
7	$2\Lambda_3$	$(0, 0, 1, 0, 0, 0)$	54	490
7	$2\Lambda_4$	$(0, 0, 0, 1, 0, 0)$	54	490

Note that the (nonzero) first eigenvalue of Δ_Σ is $(p-1)(p+2)/2 = \dim \Sigma$.

By using these results, we determine all $\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$ as follows :

If $p \geq 8$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
$(p-1)(p+2)/2$	1	$p(p+1)$
$(p-2)(p+2)$	$\frac{\sqrt{(p(p+1)/2-2)^2+4(p+2)(p-2)}-(p(p+1)/2-2)}{2}$	$\frac{2}{3}\frac{p(p-1)}{2}\frac{(p+1)p}{2}$
$p(p+1)$	2	$\frac{(p-1)p^2(p+3)}{4}$

If $p = 7$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
27	1	56
45	$\sqrt{214} - 13$	392
54	$\sqrt{223} - 13$	980
56	2	735

If $p = 6$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
20	1	42
32	$(\sqrt{489} - 19)/2$	210
36	$(\sqrt{505} - 19)/2$	175
42	2	405

If $p = 5$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
14	1	30
21	$(\sqrt{253} - 13)/2$	100
30	2	200

If $p = 4$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
9	1	20
12	$2(\sqrt{7}-2)$	20
20	2	84

The case $\Sigma = SU(2p)/Sp(p)$: In this case note that $m-1 = \dim \Sigma = (p-1)(2p+1)$, $m = 2p^2 - p = p(2p-1)$, $2m = 2p(2p-1)$ and $m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim SU(2p) = p^2(2p-3)(2p+1)$. Set

$$\begin{aligned}
&D(SU(2p)) \\
&=\Big\{\sum_{i=1}^{2p} m_i\varepsilon_i \mid \sum_{i=1}^{2p} m_i = 0, 0 \le m_i - m_{i+1} \in \mathbf{Z}\ (i = 1, 2, \ldots, 2p-1)\Big\} \\
&=\Big\{\sum_{i=1}^{2p-1} k_i\Lambda_i \mid 0 \le k_i \in \mathbf{Z}\ (i = 1, 2, \ldots, 2p-1)\Big\}.
\end{aligned} \tag{2.14}$$

Here $k_i = m_i - m_{i+1}$ $(i = 1, \cdots, 2p-1)$ and $\{\Lambda_i \mid i = 1, \cdots, 2p-1\}$ is the fundamental weight system of $SU(2p)$ defined by

$$\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{p}\sum_{j=1}^{2p}\varepsilon_j \quad (i = 1, 2, \cdots, 2p-1).$$

Now we define $f_i \in \mathbf{R}^{2p}$ by

$$f_i := \frac{1}{\sqrt{2}}(\varepsilon_{2i-1} + \varepsilon_{2i}) \quad (i = 1, 2, \cdots, p-1, p).$$

The subset $D(SU(2p), Sp(p)) \subset D(SU(2p))$ is defined by

$$\begin{aligned}
&D(SU(2p), Sp(p)) \\
&=\{\sqrt{2}\sum_{i=1}^{p} m_i f_i \mid \sum_{i=1}^{p} m_i = 0, 0 \le m_i - m_{i+1} \in \mathbf{Z}(i = 1, 2, \ldots, p-1)\} \\
&=\{\sum_{i=1}^{p-1} k_i M_i \mid 0 \le k_i \in \mathbf{Z}(i = 1, 2, \ldots, p-1)\}.
\end{aligned} \tag{2.15}$$

Here $k_i = m_i - m_{i+1}$ $(i = 1, \cdots, p-1)$ and $\{M_i \mid i = 1, \cdots, p-1\}$ is the fundamental weight system of $(SU(2p), Sp(p))$ defined by $M_i = \Lambda_{2i}$. We know that there is a bijective correspondence between $D(SU(2p), Sp(p))$ and the complete set of all inequivalent spherical representations of the compact symmetric pair $(SU(2p), Sp(p))$. Then for each $\Lambda = \sqrt{2}\sum_{i=1}^{p} m_i f_i \in D(SU(2p), Sp(p))$ we have

$$-a_\Lambda = 2\sum_{i=1}^{p}(m_i)^2 - 8\sum_{i=1}^{p} i m_i \tag{2.16}$$

and and the corresponding eigenvalue of Δ_Σ is given by

$$\lambda = (-a_\Lambda)\frac{1}{4p}C^{-1} = (-a_\Lambda)\frac{1}{4p}\cdot 2p^2 = (-a_\Lambda)\frac{p}{2} = pQ \tag{2.17}$$

because of $C = \frac{2}{p^2c} = \frac{1}{2p^2}$ by [1, p594]. Here Q is a function defined in Lemma 2.1. The multiplicity $m(\lambda)$, i.e. the dimension of the eigenspace, for the eigenvalue λ of the Laplacian Δ_Σ is equal to

$$m(\lambda) = \sum_{\Lambda\in D(SU(2p),Sp(p)),\lambda=-a_\Lambda} d_\Lambda.$$

First we consider the case $p = 3$. Then (2.16) becomes

$$-a_\Lambda = \frac{4}{3}(k_1^2 + k_1k_2 + k_2^2) + 8(k_1 + k_2). \tag{2.18}$$

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, the $(-a_\Lambda)\cdot\frac{3}{2} = \frac{28}{3}\frac{3}{2} = 14$ and $d_\Lambda = 15$.
(2) If $(k_1, k_2) = (1, 1)$, the $(-a_\Lambda)\cdot\frac{3}{2} = 20\cdot\frac{3}{2} = 30$ and $d_\Lambda = 189$.
(3) If $(k_1, k_2) =$ otherwise , then $(-a_\Lambda)\cdot\frac{3}{2} \geq 32 > 30$.

Thus all eigenvales λ and their multiplicitity $m(\lambda)$ of Δ_Σ between 0 and $2m = 30$ are determined as follows :

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
14	1	30
30	2	189

Hence we have $N_\Sigma(2) = m_\Sigma(0)+m_\Sigma(1)+m_\Sigma(2)$, $m_\Sigma(0) = 1 = b^0(\Sigma)$, $m_\Sigma(1) = 30 = 2m$, $m_\Sigma(2) = 189$. On the other hand, $m^2 - 1 - \dim G_\Sigma = 15^2 - 1 - (36 - 1) = 189$. Therefore we conclude that s-ind$(C) = 0$.

Next we treat the case $p \geq 4$. By Lemma 2.1 we obtain the following table of all $\Lambda \in D(SU(2p), Sp(p)))$ corresponding to eigenvalues $\lambda \leq 2m = 2p(2p - 1)$.

p	Λ	$(k_1, \cdots, k_{p-1})$	$\lambda = pQ$	d_Λ
≥ 3	$\Lambda_2 + \Lambda_{2p-2}$	$(1, 0, \cdots, 0, 1)$	$2p(2p - 1)$	$p^2(2p - 3)(2p + 1)$
≥ 3	Λ_2	$(1, 0, \cdots, 0, 0)$	$(2p + 1)(p - 1)$	$p(2p - 1)$
≥ 3	Λ_{2p-2}	$(0, 0, \cdots, 0, 1)$	$(2p + 1)(p - 1)$	$p(2p - 1)$
4	Λ_4	$(0, 1, 0)$	36	70
≥ 5	Λ_4	$(0, 1, 0, \cdots, 0)$	$2(2p + 1)(p - 2)$	$\frac{2p(2p-1)(2p-2)(2p-3)}{24}$
≥ 5	$\Lambda_{2(p-2)}$	$(0, \cdots, 0, 1, 0)$	$2(2p + 1)(p - 2)$	$\frac{2p(2p-1)(2p-2)(2p-3)}{24}$
6	Λ_6	$(0, 0, 1, 0, 0)$	117	924
7	Λ_6	$(0, 0, 1, 0, 0, 0)$	180	3003
7	Λ_8	$(0, 0, 0, 1, 0, 0)$	180	3003

Note that the (nonzero) first eigenvalue of Δ_Σ is $(2p + 1)(p - 1) = \dim \Sigma$.

By using these results, we determine all $\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$ as follows :

If $p \geq 8$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
$(2p+1)(p-1)$	1	$2p(2p-1)$
$2(2p+1)(p-2)$	$\frac{\sqrt{(p(2p-1)-2)^2+8(2p+1)(p-2)}-(p(2p-1)-2)}{2}$	$\frac{2p(2p-1)(2p-2)(2p-3)}{12}$
$2p(2p-1)$	2	$p^2(2p-3)(2p+1)$

If $p=7$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
90	1	182
150	$(\sqrt{8521}-89)/2$	2002
180	$(\sqrt{8641}-89)/2$	6006
182	2	8085

If $p=6$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
65	1	132
104	$\sqrt{1128}-32$	990
117	$\sqrt{1141}-32$	924
132	2	4212

If $p=5$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
44	1	90
66	$(\sqrt{253}-13)/2$	420
90	2	1925

If $p=4$, then we have

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0,2]$	$m_\Sigma(\alpha)$
0	0	1
27	1	56
36	$\sqrt{205}-13$	70
56	2	720

The case $\Sigma = E_6/F_4$: In this case $m-1 = \dim \Sigma = 26$, $m = 27$, $2m = 54$ and $m^2 - 1 - \dim G_\Sigma = m^2 - 1 - \dim E_6 = 27^2 - 1 - 78 = 650$. Let $\{M_1, M_2\}$ be the fundamental weight system of (E_6, F_4) defined by

$$M_1 = \Lambda_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6),$$
$$M_2 = \Lambda_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5,$$

where $\{\varepsilon_i \mid i = 1, \cdots, 8\}$ denotes the standard orthonormal basis of $\mathbf{R}^8$ and $\{\Lambda_i \mid i = 1, \cdots, 6\}$ denotes the fundamental weight system of E_6 (cf.[4],[1, p601]). Set

$$(2.19)\qquad \begin{aligned} &D(E_6, F_4)\\ =&\{k_1M_1 + k_2M_2 \mid k_1, k_2 \in \mathbf{Z}, k_1 \geq 0, k_2 \geq 0\}. \end{aligned}$$

Then for each $\Lambda = k_1M_1 + k_2M_2 \in D(E_6, F_4)$ we have

$$(2.20)\qquad -a_\Lambda = 4k_1(\frac{1}{3}k_1 + 4) + \frac{4}{3}k_1k_2 + 4k_2(\frac{1}{3}k_2 + 4).$$

and the corresponding eigenvalue of Δ_Σ is given by

$$(2.21)\qquad \lambda = (-a_\Lambda)\frac{1}{24}C^{-1} = (-a_\Lambda)\frac{1}{24} \cdot 36 = (-a_\Lambda)\frac{3}{2}$$

because of $C = \frac{1}{9c} = \frac{1}{36}$ by [1, p594]. Thus we determine all $\Lambda \in D(E_6, F_4)$ corresponding to eigenvalues $\lambda \leq 2m = 54$ and their multiplicities d_Λ (cf.[13]) as follows :

(1) If $(k_1, k_2) = (1, 0)$ or $(0, 1)$, then we have $\lambda = (-a_\Lambda) \cdot \frac{3}{2} = \frac{52}{3}\frac{3}{2} = 26$ and $d_\Lambda = 27$.
(2) If $(k_1, k_2) = (1, 1)$, then we have $\lambda = (-a_\Lambda) \cdot \frac{3}{2} = 36\frac{3}{2} = 54$ and $d_\Lambda = 650$.
(3) If $(k_1, k_2) =$ otherwise , then we have $\lambda = (-a_\Lambda) \cdot \frac{3}{2} \geq 56 > 54$.

Thus all eigenvalues λ and their multiplicities of Δ_Σ between 0 and $2m = 12$ are determined as follows :

λ	$\alpha \in \mathcal{D}_\Sigma \cap [0, 2]$	$m_\Sigma(\alpha)$
0	0	1
26	1	54
54	2	650

Thus we obtain $N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$, $m_\Sigma(0) = 1 = b^0(\Sigma)$, $m_\Sigma(1) = 54 = 2m$, $m_\Sigma(2) = 650 = m^2 - 1 - \dim G_\Sigma$. Hence we obtain s-ind$(C) = 0$ for $\Sigma = E_6/F_4$.

Getting together those results in each case, we conclude the following. Theorem 2.1 follows from Theorem 2.2.

Theorem 2.2. *Let $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)$ $(p \geq 3)$, E_6/F_4 (resp. $m = p^2, (p-1)(p+2)/2 + 1, (p-1)(2p+1) + 1, 27$) be an $(m-1)$-dimensional minimal Legendrian submanifold embedded in $S^{2m-1}(1)$ in the above standard way and $C = C\Sigma$ be the special Lagrangian cone in $\mathbf{C}^m$ over Σ. Then the rigidity, the Legendrian-index and the stability-index of C are described as follows:*

(1) *The equality*
$$m_\Sigma(2) = m^2 - 1 - \dim(G_\Sigma)$$
holds and hence each C is rigid.
(2) *The Legendrian-index* l-ind(C) *is equal to*
$$\text{l-ind}(C) = \text{s-ind}(C) + 2m.$$
(3) *The stability-index* s-ind(C) *is given as in the following table:*

	$SU(p)$	$SU(p)/SO(p)$	$SU(2p)/Sp(p)$	E_6/F_4
$p \geq 8$	$\frac{p^2(p-1)^2}{2}$	$\frac{p^2(p-1)(p+1)}{6}$	$\frac{2p(2p-1)(2p-2)(2p-3)}{12}$	-
$p = 7$	3332	1372	8008	-
$p = 6$	850	385	1914	-
$p = 5$	200	100	420	-
$p = 4$	36	20	70	-
$p = 3$	0	0	0	0

Remark. In [1] it was shown that for each $\Sigma = SU(p)$, $SU(p)/SO(p)$, $SU(2p)/Sp(p)$, E_6/F_4, the image $\pi(\Sigma) = SU(p)/\mathbf{Z}_p$, $SU(p)/SO(p)\mathbf{Z}_p$, $SU(2p)/Sp(p)\mathbf{Z}_{2p}$, $E_6/F_4\mathbf{Z}_3$ by the projection of the Hopf fibration is a Hamiltonian stable minimal Lagrangian submanifold embedded in a complex projective space.

And by using the formula (1.1) we also see the following.

Theorem 2.3. *In each case $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)$ $(p \geq 4)$, $C' = C\Sigma \setminus \{0\}$ has nonzero homogeneous harmonic function of order α for some α with $1 < \alpha < 2$ and there is no nonzero homogeneous harmonic function on C' of order α for any α with $0 < \alpha < 1$.*

3. Stability-index of a special Lagrangian cone in $\mathbf{C}^4$ over a minimal Legendrian $SU(2)$-orbit

In this section we mention about the stability and the rigidity of a certain special Lagrangian cone over a minimal Legendrian $SU(2)$-orbit in $\mathbf{C}^4$. This example was also treated in [9, Example 5.7].

Let V_3 be the complex vector space of all complex homogeneous polynomials with two variables z_1,z_2 of degree 3. We equip V_3 with the standard Hermitian inner product such that

$$\{v_k = \frac{1}{\sqrt{k!(3-k)!}} z_1^{3-k} z_2^k \mid k = 0, 1, 2, 3\}$$

is a unitary basis of $V_3 \cong \mathbf{C}^4 \cong \mathbf{R}^8$. We know that V_3 is an irreducible unitary representation of $SU(2)$. Now we consider the orbit of $SU(2)$ through $w = \frac{1}{\sqrt{2}}(v_0 + v_3)$. Then the orbit $\Sigma = \rho_3(SU(2))w \subset S^7(1)$ is a 3-dimensional minimal Legendrian submanifold embedded in $S^7(1)$.

Theorem 3.1. *The special Lagrangian cone C in $\mathbf{C}^4$ over the minimal Legedrian orbit $\Sigma = \rho_3(SU(2))(w)$ is not Legendrian stable, and hence not stable. Its stability-index and Legendrian-index of C are given by*

$$\text{s-ind(C)} = 10 \quad \textit{and} \quad \text{l-ind(C)} = 11 (= 8 + 3).$$

Moreover, Σ satisfies

$$m_\Sigma(2) = 19 > m^2 - 1 - \dim SU(2) = 12$$

and hence C is not rigid.

We shall calculate all the eigenvalues and thier multiplicities of the Laplacian of the $SU(2)$-orbit $\Sigma = \rho_3(SU(2))(w)$ by the method used in [15].

Let $G = SU(2)$ and $\mathfrak{g} = su(2)$. Let $\{E_1, E_2, E_3\}$ be a basis of $\mathfrak{g} = su(2)$ defined by

$$E_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For each nonnegative integer n, let (V_n, ρ_n) be an $(n+1)$-dimensional irreducible unitary representation of $G = SU(2)$ as follows: Let V_n denote a complex vector space of all complex homogeneous polynomials with two variables z_1,z_2 of degree n and $\rho_n : SU(2) \longrightarrow U(V_n)$ is defined as

$$\left(\rho_n \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} f\right)(z_1, z_2) = f((z_1, z_2)\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}). \tag{3.1}$$

Here set

$$v_k^{(n)} := \frac{1}{\sqrt{k!(n-k)!}} z_1^{n-k} z_2^k$$

for each $k = 0, 1, \ldots, n$ and the standard Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$ of V_n invariant under ρ_n is defined such that $\{v_0^{(n)}, \ldots, v_n^{(n)}\}$ is a unitary basis of V^n. Then the differential $d\rho_n$ of the representation ρ_n is given by

$$(d\rho_n\,(X)\,f)(z_1, z_2) = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right) {}^t X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.2}$$

If we denote by $\mathcal{D}(SU(2))$ the complete set of all inequivalent irreducible unitary representations of $SU(2)$, then we know

$$\mathcal{D}(SU(2)) = \{(V_n, \rho_n) \mid n \in \mathbf{Z}, n \geq 0\}.$$

In $V_3 \cong \mathbf{C}^4 \cong \mathbf{R}^8$ $(n = 3)$, we use the unitary basis $v_0 = v_0^{(3)}$, $v_1 = v_1^{(3)}$, $v_2 = v_2^{(3)}$, $v_3 = v_3^{(3)}$. Then the orbit $\Sigma = \rho_3(SU(2))w$ of $SU(2)$ through a point

$$w = \frac{1}{\sqrt{2}}(v_0 + v_3) \in S^7(1) = \{v \in V_3 \mid \|v\| = 1\}.$$

is a 3-dimensional compact minimal Legendrian submanifold embedded in $S^7(1)$. We can see that it is a unique minimal Legendrian orbit on $S^7(1)$ under ρ_3. Thus the minimal cone over $\Sigma = \rho_3(SU(2))w$ is a special Lagrangian cone in $\mathbf{C}^4$. Then $\{\frac{1}{3}E_1, \frac{1}{\sqrt{3}}E_2, \frac{1}{\sqrt{3}}E_3\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the induced metric from the orbit $\rho_3(SU(2))v \subset \mathbf{C}^4$. We denote by Δ_Σ the Laplacian of $\Sigma = G/K$ with respect to the induced metric acting smooth fuctions on G/K. The isotropy subgroup

$$K := \{A \in G \mid \rho_3(A)w = w\} \tag{3.3}$$

of $G = SU(2)$ at $w \in V_3$ is a cyclic subgroup $\mathbf{Z}_3$ of order 3 consisting of the following elements

$$\tag{3.4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^{\sqrt{-1}\frac{2\pi}{3}} & 0 \\ 0 & e^{-\sqrt{-1}\frac{2\pi}{3}} \end{pmatrix}, \quad \begin{pmatrix} e^{-\sqrt{-1}\frac{2\pi}{3}} & 0 \\ 0 & e^{\sqrt{-1}\frac{2\pi}{3}} \end{pmatrix}$$

which is the fundamental group of $\Sigma \cong G/K$.

For each nonnegative integer n, we define a vector subspace $(V_n)_K$ of V_n by

$$\tag{3.5} (V_n)_K := \{v \in V_n \mid \rho_n(A)v = v \text{ for each } A \in K\}.$$

Then by direct computations we have

Lemma 3.1.

(1) *In case $n = 2\ell$:*
If we set $\ell = 3p + r$ for $p \in \mathbf{Z}$ with $p \geq 0$ and $r \in \mathbf{Z}$ with $0 \leq r < 3$, then $(V_n)_K$ is spanned by

$$\{v_k^{(n)} \mid k = \ell + 3j \quad (j = -p, \cdots, -1, 0, 1, \cdots, p)\}.$$

(2) *In case $n = 2\ell + 1$:*
If $2\ell + 1 = 3p$ for $p \in \mathbf{Z}$, then $(V_n)_K$ is spanned by

$$\{v_k^{(n)} \mid k = 3j \ (j = 0, 1, \cdots, p)\}.$$

If $2\ell + 1 = 3p + 1$ for $p \in \mathbf{Z}$, then $(V_n)_K$ is spanned by

$$\{v_k^{(n)} \mid k = 3j - 1 \ (j = 1, 2, \cdots, p)\}.$$

If $2\ell + 1 = 3p + 2$ for $p \in \mathbf{Z}$, then $(V_n)_K$ is spanned by

$$\{v_k^{(n)} \mid k = 3j + 1 \ (j = 0, 1, \cdots, p)\}.$$

By Peter-Weyl's theorem we know

$$\tag{3.6} C^\infty(G/K) = \bigoplus_{n \in \mathbf{Z}, n \geq 0} (V_n)_K^* \otimes V_n.$$

Here each $v \in (V_n)_K$ and each $u \in V_n$ corresponds to $f \in C^\infty(G/K)$ defined by

$$f(aK) := \langle\langle \rho_n(a)v, u \rangle\rangle \quad (aK \in G/K).$$

Then we have

$$\tag{3.7} \begin{aligned} &(\Delta_\Sigma f)(aK) \\ =&\langle\langle \rho_n(a) \left((d\rho_n(\frac{1}{3}E_1))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_2))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_3))^2 \right) v, u \rangle\rangle. \end{aligned}$$

By direct computations we have the following lemmas.

Lemma 3.2.

$$\tag{3.8} \begin{aligned} &\left((d\rho_n(\frac{1}{3}E_1))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_2))^2 + (d\rho_n(\frac{1}{\sqrt{3}}E_3))^2 \right) v_k^{(n)} \\ &= - \Big\{ \frac{1}{9}(n - 2k)^2 + \frac{2}{3}((k + 1)(n - k) + k(n - k + 1)) \Big\} v_k^{(n)}. \end{aligned}$$

Lemma 3.3. *All eigenvalues and their multiplicities of Δ_Σ are given as follows: Let $n \in \mathbf{Z}$ with $n \geq 0$.*

(1) *In case $n = 2\ell$, if we set $\ell = 3p + r$ with nonnegative $p, r \in \mathbf{Z}$ and $0 \leq r < 3$, Δ_Σ has eigenvalues*

$$\frac{4}{3}\ell(\ell+1) - 8j^2 \quad (j = -p, \cdots, -1, 0, 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 1$.

(2) *In case $n = 2\ell + 1$, if $2\ell + 1 = 3p$ for an integer $p \geq 1$, then Δ_Σ has eigenvalues*

$$(p-2j)^2 + 2((3j+1)(p-j) + j(3p-3j+1)) \quad (j = 0, 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 2$.

(3) *In case $n = 2\ell + 1$, if $2\ell + 1 = 3p + 1$ for an integer $p \geq 2$, then Δ_Σ has eigenvalues*

$$(p-2j+1)^2 + 2(j(3p-3j+2) + (3j-1)(p-j+1)) \quad (j = 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 2$.

(4) *In case $n = 2\ell + 1$, if $2\ell + 1 = 3p + 2$ for an integer $p \geq 1$, then Δ_Σ has eigenvalues*

$$(p-2j)^2 + \frac{2}{3}((3j+2)(3p-3j+1) + (3j+1)(3p-3j+2)) \quad (j = 0, 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 2$.

From Lemma 3.3 all eigenvalues of Δ_Σ not greater than $\lambda \leq 2m = 8$ and their multiplicities are given as follows :

(1) For $n = 2$, $\ell = 1$ and $j = 0$, the eigenvalue is $\frac{8}{3}$ ($\alpha = \frac{\sqrt{33}}{3} - 1$) and its multiplicity is 3.
(2) For $n = 3$, $\ell = 1, p = 1$ and $j = 0$, the eigenvalue is 3 ($\alpha = 1$) and its multiplicity is 4.
(3) For $n = 3$, $\ell = 1, p = 1$ and $j = p$, the eigenvalue is 3 ($\alpha = 1$) and its multiplicity is 4.
(4) For $n = 4$, $\ell = 2, p = 0$ and $j = 0$, the eigenvalue is 8 ($\alpha = 2$) and its multiplicity is 5.
(5) For $n = 6$, $\ell = 3, p = 1$ and $j = -1$, the eigenvalue is 8 ($\alpha = 2$) and its multiplicity is 7.
(6) For $n = 6$, $\ell = 3, p = 1$ and $j = 1$, the eigenvalue is 8 ($\alpha = 2$) and its multiplicity is 7.
(7) Otherwise all other eigenvalues are greater than 8.

Thus we have

$$m_\Sigma(0) = 1, \quad m_\Sigma(\frac{\sqrt{33}}{3} - 1) = 3,$$
$$m_\Sigma(1) = 4 + 4 = 8, \quad m_\Sigma(2) = 5 + 7 + 7 = 19,$$

and

$$N_\Sigma(2) = m_\Sigma(0) + m_\Sigma(\frac{\sqrt{33}}{3} - 1) + m_\Sigma(1) + m_\Sigma(2) = 31.$$

Therefore we obtain

$$\text{s-ind}(C) = N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G_\Sigma = 10.$$

and

$$\text{l-ind}(C) = m_\Sigma(\frac{\sqrt{33}}{3} - 1) + m_\Sigma(1) = 11 > 8.$$

and hence C is not Legendrian-stable. And we obtain

$$m_\Sigma(2) = 19 > 12 = 4^2 - 1 - \dim SU(3) = m^2 - 1 - \dim G_\Sigma.$$

and hence C is not rigid. Therefore we obtain Theorem 3.1.

And by using the formula (1.1) we also see the following.

Theorem 3.2. *For this minimal Legendirian orbit $\Sigma = \rho_3(SU(2))w$, $C' = C \setminus \{0\}$ has nonzero homogeneous harmonic function of order α for some α with $0 < \alpha < 1$ and there is no nonzero homogeneous harmonic function on C' of order α for any α with $1 < \alpha < 2$.*

Next we consider the Hopf fibration $\pi : S^7(1) \longrightarrow \mathbf{C}P^3$ from $S^7(1) \subset V_3 \cong \mathbf{C}^4$ onto the 3-dimensional complex projective space $\mathbf{C}P^3$ with the Fubini-Study metric of constant holomorphic sectional curvature 4. We denote also by ρ_3 the action of $SU(2)$ on $\mathbf{C}P^3$ induced by π from the represenation ρ_3 of $SU(2)$ on $V_3 \cong \mathbf{C}^4$. By the projection of the minimal Legendrian orbit $\rho_3(SU(2))w$, we obtain a minimal Lagrangian orbit $L = \rho_3(SU(2))[w]$ on $\mathbf{C}P^3$ through $[w] = \mathbf{C}w$. It was also treated in [5] from the viewpoint of momentum maps. Then the isotropy subgroup

$$K' := \{A \in SU(2) \mid \rho_3(A)[w] = [w]\} \tag{3.9}$$

of $SU(2)$ at $[w] \in \mathbf{C}P^3$ is a finite subgroup of order 12 consisting of the following elements

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\bar{b} \\ b & 0 \end{pmatrix} \tag{3.10}$$

where $a, b \in \mathbf{C}$ with $|a| = |b| = 1$ and $a^6 = 1$, $b^6 = -1$. Let

$$(V_n)_{K'} := \{v \in V_m \mid \rho_m(A)v = v \text{ for each } A \in K'\}. \tag{3.11}$$

Note that $K \subset K'$ and thus $(V_n)_{K'} \subset (V_n)_K$. Then by checking the results of Lemma 3.1 on $(V_n)_K$ we can show

Lemma 3.4. (a) $(V_n)_{K'} \neq \{0\}$ *if and only if $n = 2\ell$ for some integer $\ell \in \mathbf{Z}$ satisfying the condition that ℓ is odd with $\ell \geq 3$, or that ℓ is even with $\ell \geq 0$,*

(b) *If $n = 2\ell$ and ℓ is odd with $\ell \geq 3$, setting $\ell = 3p + r$ for $0 \leq p \in \mathbf{Z}$ and $0 \leq r < 3$, then*

$$\{v_{\ell+3j}^{(2\ell)} - v_{\ell-3j}^{(2\ell)} \mid j = 1, \cdots, p\}$$

is a basis of $(V_n)_{K'}$.

(c) *If $n = 2\ell$ and ℓ is even with $\ell \geq 0$, setting $\ell = 3p + r$ for $0 \leq p \in \mathbf{Z}$ and $0 \leq r < 3$, then*

$$\{v_{\ell+3j}^{(2\ell)} + v_{\ell-3j}^{(2\ell)} \mid j = 0, 1, \cdots, p\}$$

is a basis of $(V_n)_{K'}$.

Now by using Lemma 3.2 we can determine all eigenvalues for the Laplacian Δ' of L on functions.

Lemma 3.5. *All eigenvalues and their multiplicities of Δ' are given as follows: Let $n = 2\ell$ for $\ell \in \mathbf{Z}$ with ell ≥ 0.*

(1) *In the case when ℓ is odd and $\ell \geq 3$, if we set $\ell = 3p+r$ with nonnegative $p, r \in \mathbf{Z}$ and $0 \leq r < 3$, Δ' has eigenvalues*

$$\frac{4}{3}\ell(\ell + 1) - 8j^2 \quad (j = 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 1$.

(2) *In the case when ℓ is even and $\ell \geq 0$, if we set $\ell = 3p+r$ with nonnegative $p, r \in \mathbf{Z}$ and $0 \leq r < 3$, Δ' has eigenvalues*

$$\frac{4}{3}\ell(\ell + 1) - 8j^2 \quad (j = 0, 1, \cdots, p)$$

and its multiplicity is $n + 1 = 2\ell + 1$.

By Lemma 3.5 we can determine the first eigenvalue of Δ' and its multiplicity as follows:

Lemma 3.6. (1) *If $n = 4$, $\ell = 2, p = 0$ and $j = 0$, then the eigenvalue is* 8 *and its multiplicity is* 5.

(2) *If $n = 6$, $\ell = 3, p = 1$ and $j = 1$, then the eigenvalue is* 8 *and its multiplicity is* 7.

(3) *Otherwise all other eigenvalues are greater than* 8.

Hence we obtain that the first eigenvalue of Δ' is 8 and its multiplicity is $12 = 4^2 - 1 - \dim(SU(2))$. Therefore we conclude

Corollary 3.1. *$\pi(\Sigma) = \rho_3(SU(2))[w]$ is a 3-dimensional compact Hamiltonian stable minimal Lagrangian submanifold embedded in $\mathbf{C}P^3$ which does not have parallel second fundamental form. Moreover its null space is exactly the span of the normal projections of Killing vector fields on $\mathbf{C}P^3$.*

Remark. This example gives a negative answer to the second problem in [1, p506]. Very recently it was also obtained independently by Lucio Bedulli and Anna Gori in their paper : A Hamiltonian stable minimal Lagrangian submanifolds of projective spaces with non-parallel second fundamental form.

References

[1] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces*, Tohoku Math. J. 55 (2003), 583–610.

[2] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain H-minimal Lagrangian submanifolds and related problems*, Surikaisekikenkyusho Kokyuroku 1292 (2002), General Study on Riemannian submanifolds, 2002, RIMS, Kyoto University, Kyoto Japan, 72–93.

[3] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain symmetric R-spaces embedded in complex Euclidean spaces*, preprint, Tokyo Metropolitan University, 2002.

[4] N. Bourbaki, *Elements de mathematique. Fasc. XXXIV. Groupes et algebres de Lie*, Actualites Scientifiques et Industrielles, No. 1337, Hermann, Paris 1968.

[5] R. Chiang, *New Lagrangian submanifolds of* $\mathbf{C}P^n$, Int. Math. Res. Not. *45* (2004), 2437–2441.

[6] M. Haskins, *Special Lagrangian cones*, Amer. J. Math. *126* (2004), 845–871.

[7] M. Haskins, *The geometric complexity of special Lagrangian* T^2*-cones*, Invent. math. 157 (2004), 11–70.

[8] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Mathematica *148* (1982), 47–157.

[9] D. Joyce, *Special Lagrangian m-folds in* $\mathbf{C}^m$ *with symmetries*, Duke Math. J. 115 (2002), 1–51. math.DG/0008021.

[10] D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity*, Ann. Global Anal. Geom. 25 (2004), 201–251. math.DG/0211294.

[11] D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces*, Ann. Global Anal. Geom. 25 (2004), 301–352. math.DG/0211295.

[12] D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, J. Differential Geom. 63 (2003), 279–347. math.DG/0303272.

[13] W. G. McKay and J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 69. Marcel Dekker, Inc., New York-Basel, 1981.

[14] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. *6* (1998), 705–747.

[15] H. Muto, Y. Ohnita and H. Urakawa, *Homogeneous minimal hypersurfaces in the unit spheres and the first eigenvalues of their Laplacian*, Tohoku Math. J. *36* (1984), 243–267.

[16] Y. Ohnita, *The first standard minimal immersions of compact irreducible symmetric spaces*, Differential Geometry of Submanifolds, edited by Katsuei Kenmotsu, Lecture Notes in Mathematics 1090, Springer-Verlag, 1984, 37–49.

[17] Y.Ohnita, *Satbility and Rigidity of Certain Special Lagrangian Cones*, Surikaisekikenkyusho Kokyuroku 1460 (2005), Differential Geometry and Submanifold Theory, RIMS, Kyoto University, Kyoto Japan, 43–52.

[18] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

[19] M. Takeuchi, *Modern Theory of Spherical Functions*, Iwanami, Tokyo, 1975, (in Japanese). *Modern Spherical Functions*, Translations of Mathematical Monographs, **135**. American Mathematical Society, Providence, RI, 1994.

Department of Mathematics Osaka City University Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, JAPAN

E-mail address: `ohnita@sci.osaka-cu.ac.jp`