FUTURE RESEARCH

The main focus of my current research is on M. Kashiwara's theory of crystals, which are combinatorial models for representations of symmetrizable Kac–Moody Lie algebras, and exploring their connections to other areas such as geometry, number theory, probability theory, and mathematical physics. One important connection is to statistical mechanics, where crystals arise naturally in the study of integrable systems, such as solvable lattice models and quantum spin chains. I am currently trying to generalize crystals, in particular those for Kirillov–Reshetikhin (KR) modules, to Lie superalgebras and develop a super analog of the $X = M$ conjecture of G. Hatayama et al. I have also recently been looking at understanding the K-theory of the Grassmannian through various techniques with applications to probability theory.

A Lie superalgebra is a $(\mathbb{Z}/2\mathbb{Z})$-graded generalization of a Lie algebra characterized by a twisted Lie bracket introduced by V. Kac to unify bosons and fermions and encode the "supersymmetries" that are essential in modern string theory. Despite the representation theory being more complicated, there are crystals for various Lie superalgebras due to the work of various authors such as G. Benkart, S.-J. Kang, and M. Kashiwara. There are known numerous parallels to the theory of KR crystals for $\widehat{\mathfrak{gl}}(m|n)$, such as Weyl modules, fusion construction, box-ball systems and a super Korteweg–de Vries (KdV) equation, super T-systems, etc. Therefore, there should be a deep and rich theory of *KR supercrystals*

Problem 1. *Construct a theory of KR supercrystals, super rigged configurations, and super $X = M$.*

The study has recently been initiated by J.-H. Kwon and M. Okado with the first example of KR supercrystals for $\widehat{\mathfrak{gl}}(m|n)$. I expect that an answer would lead to significant developments in cluster superalgebras, developing a theory of super quiver varieties, and a crystal theoretic construction of the super box-ball system of K. Hikami and R. Inoue and generalizations to other types. Two of my students have obtained results for a super soliton cellular automata using the Kwon–Okado KR supercrystals and are currently writing their paper. B. Salisbury and I also developed a model for $B(\infty)$ for the queer Lie superalgebra $\mathfrak{q}_n$, which we conjecture to the the crystal basis for the corresponding Verma module.

The other major project I have been working on recently is related to the K-theory of Schubert varieties, which comes from the study of the Grassmannian: the set of k-dimensional subspaces of $\mathbb{C}^n$. The K-theory ring is determined by *(symmetric) Grothendieck polynomials* G_λ, which have a combinatorial interpretation by *set-valued (semistandard)* tableaux of the (partition) shape λ due to A. Buch. There is also a "partial" version called *Lascoux polynomials* that were conjectured to have combinatorial interpretations, only one of which has been proven by my work with V. Buciumas and K. Weber. I am currently working on another interpretation due to C. Ross and A. Yong, with the only partial results due to myself and O. Pechenik. The work with C. Monical and O. Pechenik suggests a Lie algebra action on the corresponding geometric objects and a generalization of crystals, which we call (weak) K-crystals, to categorify the K-theory ring of the Grassmannian.

Problem 2. *Show that the (weak) K-crystal structure exists for all partitions and the corresponding characters give a categorification of Grothendieck polynomials and Lascoux polynomials.*

My recent work with K. Motegi gives a probabilistic interpretation of a refined version of the dual basis to Grothendieck polynomials g_λ as a random matrix process called *last-passage percolation*. Our work suggests the following problem.

Problem 3. *Determine probabilistic interpretations for Grothendieck polynomials and other related polynomials.*

Another direction to generalize my results would be to "melt" the crystal structure to a positive temperature $q > 0$ model (rather than a $q = 0$ model for the quantum group $U_q(\mathfrak{g})$). This is well-understood on the side of the random matrix models, but for g_λ, we would need to find a replacement for the combinatorics: reverse plane partitions. A natural analog is using an affine algebraic variety that contains an analog of the crystal structure called a geometric crystal introduced by A. Berenstein and D. Kazhdan.

Problem 4. *Determine a geometric crystal structure on reverse plane partitions. Furthermore, construct a positive temperature version of the bijection between reverse plane partitions and the last-passage percolation process, which should be a geometric crystal isomorphism.*

Let me also quickly describe three other projects I am currently working on. An open question regarding KR crystals is to give a uniform model for all KR crystals, which is currently done type-by-type. I am currently writing up a result with I. S. Jang where we give a uniform model in the case of special KR crystals, which are indecomposible as classical crystals. S.-j. Oh and I are continuing to work on proving the monoidal categorification conjecture of D. Hernandez and B. Leclerc by proving the denominator formulas and Dorey's rule for KR modules to better understand the category of finite-dimensional modules of affine quantum groups. V. Buciumas and I recently developed a vertex model interpretation for Demazure characters and atoms for the symplectic and odd orthogonal Lie groups, and we are looking into expand this model to the corresponding K-theory rings.