# これからの研究計画（英訳） 中村徹也

My research interest is CMC 1 surfaces in $\mathbf{H}^3$, especially those with smooth ends. I would like to study them by using pairs of spinors. More precisely, I am interested in (1) Construction of Surfaces, (2)Description of an Integrable System, and (3)Formulation as an Variational Problem.

## (1) Construction of Surfaces

CMC 1 spheres with smooth ends whose number of ends and Willmore energies are very small are already classified. However, the moduli space of the surfaces with larger number of ends and Willmore energies have not yet been understood. For spheres with three smooth ends (trinoids), for example, it is known that the pole orders of the local spinors at the ends necessarily satisfy the triangle inequalities. It is interesting to show the (non-)sufficiency of this condition. There should exist trinoids with the symmetry of dihedral group $D_3$. Furthermore, I would like to show the existence of a real parameter family of trinoids which interpolate smooth ended ones, where the parameter should be given as the exponentials of the local spinors at each end. This would give a three ends analogue to the family of catenoid cousins.

Constructing CMC 1 tori with smooth ends is itself an interesting project. [CSN01] has proven the existence of cousins of Costa-Hoffman-Meeks tori. However, because their proof is based on an implicit method, they do not discuss the smoothness of the ends. We would like to show the existence of smoothly ended Costa-Hoffman-Meeks cousins by applying our explicit methods using pairs of spinors. Another interesting example would be those with non-zero constant Hopf differential. such a surface would be an example of constrained Willmore surface (see (3)). For such a surface, it is also known that the Schwarzian derivatives of the spinors are related to Picard potentials which come from the theory of elliptic finite gap solutions of stationary KdV hierachy.

## (2) Description of an Integrable System

The pairs $\frac{1}{\sqrt{Q}}\binom{P}{Q}$, $\frac{1}{\sqrt{Q}}\binom{p}{q}$ of spinors divided by the square root of the Hopf differential satisfy Hill's equations whose potentials are given by their Schwarzian deriatives. We would like to find an integrable system description for the symmetries of these equations, which are analogous to KdV hierarchy. We already know that $\mathrm{Sp}(\mathbb{C}^4)$ gives symmetries for the equations, thus the first step for us to take would be analizing this group action. Another hint for the integrable system description might be given by Darboux transforms which act on CMC 1 surfaces. We would like to describe Darboux transforms in terms of pairs of spinors and figure out how they are related to $\mathrm{Sp}(\mathbb{C}^4)$ transforms. With this, we would understand the symmetries of the equations more uniformly and this would help us find an integrable system description.

## (3) Formulation as an Variational Problem

Although a CMC 1 surface in $\mathbf{H}^3$ is a constrained Willmore surface, if we compactify each smooth end in $\mathbb{R}^3$, the surface is not constrained Willmore anymore in general. This is because the Hopf differential may have poles at the ends [BPP08]. Thus in order to formulate CMC 1 surfaces with smooth ends as critical points of some variational problem, we have to consider variations for the Willmore energy with some fixed point constraints. Such a formulation is definitly an interesting problem and it could shed some light on the classification of general Willmore surfaces.

# 参考文献


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