これまでの研究成果（英訳） 中村徹也

My research is mainly about constant mean curvature (CMC) 1 surfaces in the three dimensional hyperbolic space $\mathbf{H}^3$. Let $f\colon M \to \mathbf{H}^3$ be a CMC 1 conformal immersion. [Bry87] found a holomorphic represenation for this surface which is given by $f = \Phi\,\Phi^\dagger$ with a holomorphic map $\Phi\colon \widetilde{M} \to \mathrm{SL}_2(\mathbb{C})$ defined on the universal cover $\widetilde{M}$. Hereby $\Phi^\dagger$ denotes the transposed complex conjugate matrix, and we identify $\mathbf{H}^3$ with the set of 2×2 Hermitian matrices of determinant one and positive trace. Φ satisfy (1) $d\Phi$ is nowhere zero, (2) $\det(d\Phi) \equiv 0$, and (3) Φ has an SU(2) monodromy from the right. From conditions (1) and (2), we can understand Φ as a null curve immersion into a three dimensional quadric. From this perspective, we can derive [Nak22] an integral free formula

$$\Phi = \frac{1}{S_{\binom{p}{q}} - S_{\binom{P}{Q}}} \begin{pmatrix} -\begin{vmatrix} P & \nabla P \\ q & \nabla q \end{vmatrix} & \begin{vmatrix} P & \nabla P \\ p & \nabla p \end{vmatrix} \\ -\begin{vmatrix} Q & \nabla Q \\ q & \nabla q \end{vmatrix} & \begin{vmatrix} Q & \nabla Q \\ p & \nabla p \end{vmatrix} \end{pmatrix} .$$

Here we call $\binom{P}{Q}$ and $\binom{p}{q}$ (pairs of) the global and local spinors, which are defined through

$$d\Phi\,\Phi^{-1} = \binom{P}{Q}\,(\,-Q,\; P\,) \quad \text{and} \quad \Phi^{-1} d\Phi = \binom{p}{q}\,(\,-q,\; p\,)$$

$S_\bullet$ denotes the Schwarzian derivative of each pair of spinors.

Let $M = \Sigma \setminus \{p_1, \cdots, p_n\}$ be a punctured Riemann surface and assume that the spinors $\binom{P}{Q}$ and $\binom{p}{q}$ extend into each puncture p_i with poles and $\mathrm{ord}_{p_i}\binom{P}{Q} = -1$. Then the surface has smooth ends at p_i, which are by definition ends such that the immersion f extends into p_i as a smooth immersion into $\mathbb{R}^3$, where we consider $\mathbf{H}^3$ as the conformal unit ball in $\mathbb{R}^3$ known as the Poincaré model [BP09].

In terms of spinors, we get simple expressions for geometric invariants such as Hopf differntial, hyperbolic Gauß map, the number of ends, Willmore energy, etc. If Riemann surface Σ is compact and every p_i is a smooth end, the total order $\sum_i \mathrm{ord}_{p_i}\binom{P}{Q}$ of the global spinors is equal to the negative of the number of ends while the total order $\sum_i \mathrm{ord}_{p_i}\binom{p}{q}$ of the local spinors is equal to the Willmore energy multiplied with -4π. Applying this formulation, smooth ended CMC 1 spheres with Willmore energy up to 16π were classified [Nak22].

For CMC 1 spheres with three smooth ends, some necessary conditions on the pole orders of local spinors at ends were derived and some examples were constructed [Nak22]. However the sufficiency of the conditions is still to prove in our formulations.

We can gain spinors of new CMC 1 surfaces by multiplying $\mathrm{Sp}(\mathbb{C}^4)$ matrices to spinor vectors $(P, Q, p, -q)^T$ followed by proper rescaling. In general these new surfaces are only locally defined. However the new surfaces become closed again if we consier CMC 1 spheres with smooth ends. In this case, for typical $\mathrm{Sp}(\mathbb{C}^4)$ matrices, the ends of the new spheres are again smooth and the Willmore energy is unchanged. An example is a periodic deformation of catenoid cousins which interpolate a smoothly ended catenoid cousin and its dual surface. In this example, we can observe that smooth ends converge into one (smooth or non-smooth) end and again split into smooth ends. Through the deformation, the symmetry of a dihedral group is preserved [Nak22].

参考文献

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