FUTURE RESEARCH PLAN

Tsutomu Nakamura

The following are the main themes of my future research and their summaries.

1. Generic modules in large CM representation theory

For finite-dimensional algebras, large (i.e., possibly infinitely generated) representations based on pure-injective modules have been deeply studied from various viewpoints. For some class of finite-dimensional algebras (over an algebraically closed field), it is revealed that their infinitely generated indecomposable pure-injective modules are classified into three types (Puninski–Prest, 2016). One of them is the generic modules, which are finitely generated modules of finite length over their endomorphism rings. Roughly speaking, the generic modules over a finite-dimensional algebra (over an algebraically field) have a role to control the behavior of finitely generated indecomposable modules. Moreover, tameness of finite-dimensional algebras can be characterized by using generic modules (Crawley-Boevey, 1991). In this research theme, I aim at extending the theory of generic modules over finite-dimensional algebras to large Cohen–Macaulay (CM) representation theory.

2. CM hearts and big CM modules

Big CM modules are a natural generalization of finitely generated CM modules to large modules, and they are traditional in commutative algebra. As conjectured by Hochster (1975), and finally solved by André (2018), every commutative noetherian local ring has a (balanced) big CM module. In recent years, constructions and applications of big CM modules have actively been studied. On the other hand, it is still open if every commutative noetherian complete local ring has a finitely generated CM module.

In the collaboration with Michal Hrbek and Jan Šťovíček (mentioned in "Research results so far"), we introduced the notion of the CM heart $\mathcal{H}_{\mathsf{CM}}$ for a commutative noetherian ring. This is the heart of some compactly generated t-structure in the (unbounded) derived category of R, and is a locally finitely presented Grothendieck category. Every complex belonging to $\mathcal{H}_{\mathsf{CM}}$ has CM property in some sense, and in particular, every (balanced) big CM modules belongs to $\mathcal{H}_{\mathsf{CM}}$. In this research theme, I aim at discovering new phenomena of big CM modules from the viewpoint of the abelian category $\mathcal{H}_{\mathsf{CM}}$.

3. Flat cotorsion quasi-coherent sheaves over noetherian schemes

A natural generalization of the category of modules over a commutative noetherian ring is the category of quasi-coherent sheaves $\mathsf{Qcoh}\, X$ over a noetherian scheme X. It is a quite big difference that the former always has enough projective objects while the latter may not. In this sense, it is indeed fundamental that $\mathsf{Qcoh}\, X$ has enough flat objects. However, the flat objects in $\mathsf{Qcoh}\, X$ can usually not realize the derived category of $\mathsf{Qcoh}\, X$ as a homotopy category. This defect can be filled by considering the flat cotorsion quasi-coherent sheaves. That is, the derived category of quasi-coherent sheaves over a (quasi-compact semi-separated) scheme is triangulated equivalent to the homotopy category of flat cotorsion quasi-coherent sheaves, like the case of modules over a ring. Partly motivated by this fact, I am collaborating with Sergio Estrada and Ryo Kanda to study structure of flat cotorsion quasi-coherent sheaves.